Discrete Length and Area Biased Exponential Distribution

Nuri Celik [a*]

[a]*Department of Mathematics, Gebze Technical University, Turkey.*

Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Original Research Article

Abstract

The preference of the proper distribution for modelling data is often a substantial concern to researchers and practitioners. For this reason, new statistical distributions or the generalizations of well-known distributions have been proposed for flexible modeling. Weighted distributions are one of the generalization methods for these distributions. In this article, we proposed e new discrete distribution which takes length and area biased exponential distributions as an underlying distribution. Main statistical properties of these proposed distributions are obtained and real-life examples which have been used in literature are used for illustration of these distributions.

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*Corresponding author: E-mail: nuricelik@gtu.edu.tr;

1   Introduction

In recent years, it has been seen that, traditional statistical distributions have limited application areas. For this reason, some authors have proposed some generalization methods of these distributions or some other new distributions. On the other hand, by using the statistical analysis which strongly depends on the assumed probability model or distributions [1]. Discretization of continuous distribution is one of the most applied methodologies used in statistical literature. If the underlying continuous random variable X has the survival function $S(x) = 1 - F(x) = P(X > x)$ then the random variable Y (largest integer less than or equal to X) will have the discrete probability function as

$$f(y) = S_x(y) - S_x(y+1), y = 0, 1, 2, ... \tag{1.1}$$

There are many studies dealing with this discretization method. Nakagawa and Osaki [2] first proposed discrete Weibull distribution, Stein and Dattero [3] presented another discretization of Weibull distribution, Roy [4] considered discrete normal distribution, he also studied discrete Rayleigh distribution, Krishna and Pundir [5] studied discrete Burr distribution, Chakraborty and Chakravarty [6] proposed discrete gamma distribution, Jazi et al. [7] proposed discrete inverse Weibull distribution and Bakouch et al. [8] studied discrete Lindley distribution. Additionally, these proposed distributions have been applied to so many areas such as renewal theory, molecular biology, reliability, finance, value at risk problems and water management, see for example Nekoukhou et al [9], Roy [10], Lin and Guillén[11], Haas [12]and Wang et al. [13].

On the other hand, exponential distribution is one of the key distributions in statistics application and theory. Exponential distribution has a huge application area because of its lack of memory property, its constant hazard rate and its great mathematical tractability. Many generalizations of the exponential distribution are developed in recent years such as the exponentiated exponential [14, 15], generalized exponentiated moment exponential [16], extended exponentiated exponential [17], Marshall-Olkin exponential Weibull [18], Marshall-Olkin generalized exponential [19], and exponentiated moment exponential [20] distributions. In this paper, we use the weighted exponential distribution as an underlying continuous distribution.

The weighted distribution is defined as

$$f(x; \theta) = \frac{w(x) f_0(x; \theta)}{E(w(x))} \tag{1.2}$$

where $w(x)$ is any function of random variable X. Size-biased distributions are the special cases of the weighted distributions and has the form

$$f(x; \theta) = \frac{x^k f_0(x; \theta)}{\mu_k} \tag{1.3}$$

where $f_0(x; \theta)$ is the original underlying distribution and $\mu_k = E(X^k)$ is the kth moment of the random variable X. We get the length-biased and area-biased distributions for $k = 1$ and $k = 2$ respectively.

Let X be a random variable which follows length-biased exponential distribution with parameter θ, then the probability density function is given by

$$f(x; \theta) = \theta^2 x exp(-\theta x), \ x > 0 \tag{1.4}$$

and the corresponding cumulative distribution function is obtained as

$$F(x; \theta) = 1 - (\theta x + 1) exp(-\theta x), \ x > 0. \tag{1.5}$$

On the other hand, let X be a random variable which follows area-biased exponential distribution with parameter θ, then the probability density function and cumulative distribution function are given by

$$f(x;\theta) = \frac{\theta^3}{2}x^2 exp(-\theta x),\ x > 0 \tag{1.6}$$

and

$$F(x;\theta) = 1 - \left(\frac{\theta^2 x^2 + 2\theta x + 2}{2}\right)exp(-\theta x),\ x > 0 \tag{1.7}$$

respectively.

2 Discrete Length-Biased Exponential Distribution

The probability mass function of the discrete length-biased exponential (DLBE) distribution can be obtained by using Equation (1.1) and Equation (1.5) as follows:

$$P(X = x) = q^x\Big(\theta x(1 - q) + 1 - \theta q - q\Big), x = 0, 1, 2, ... \tag{2.1}$$

where $q = exp(-\theta)$. The corresponding cumulative distribution function and survival function of the DLBE distribution are

$$F(x) = 1 - q^{x+1}\Big(1 + \theta + \theta x\Big), x = 0, 1, 2, ... \tag{2.2}$$

and

$$S(x) = q^{x+1}\Big(1 + \theta + \theta x\Big), x = 0, 1, 2, ... \tag{2.3}$$

respectively.

Fig. 1 shows some possible shapes of Equation (2.1). It can be seen that the probability mass function is always unimodal.

Fig. 2 shows some possible shapes of hazard rate function $h(x) = f(x)/S(x)$ for different θ values. It can be easily seen that the failure rate is increasing with respect to x and θ values.

As a result, the DLBE distribution is unimodal, since the probability function satisfies the log-concave inequality $\big(f^2(x) > f(x+1)f(x-1)\big)$, see Keilson and Gerber, [21]. Additionally, the $DLBE$ distribution is unimodal and has a discrete increasing failure rate.

The quantile function of DLBE distribution can be obtained by solving the equation $F(Q(u)) = u$.

$$1 - u = exp\Big(-\theta(Q(u) + 1)(1 + \theta + \theta Q(u))\Big) \tag{2.4}$$

Substituting $Z(u) = -1 - \theta - \theta Q(u)$, the quantile function can be obtained as,

$$Q(u) = -\frac{W\left(e^{-1}(u - 1)\right) + \theta + 1}{\theta} \tag{2.5}$$

for $0 < u < 1$, where $W(.)$ is the Lambert W function. In particular, the median of the DLBE distribution is

$$Q(\frac{1}{2}) = -\frac{W\left(e^{-1}(1 - \frac{1}{2})\right) + \theta + 1}{\theta} \tag{2.6}$$
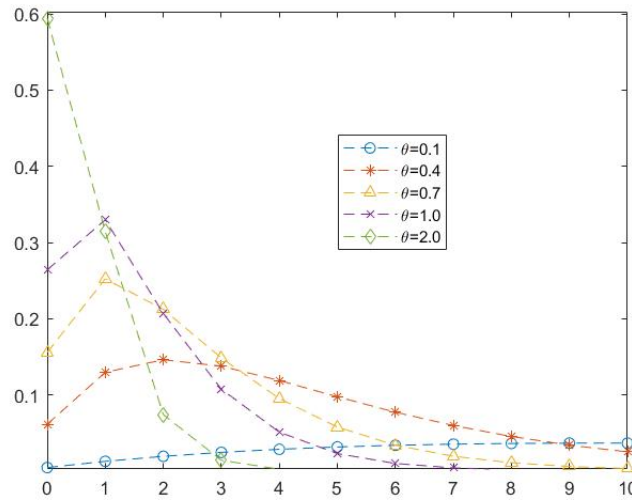
Fig. 1. Possible shapes of probability function of the DLBE distribution

The moment generating function can be found by using probability generating function $E(t^X)$ as:

$$M_x(t) = (1 - \theta q - q)\frac{1}{1 - qe^t} + \theta(1 - q)\frac{qe^t}{(1 - qe^t)^2} \tag{2.7}$$

Consequently, the first four moment of $DLBE$ distribution can be found by using the derivatives of moment generating function obtained in Equation (2.7).

$$E(X) = \frac{q(1 - \theta q - q) - \theta q(q - 1) + 2\theta q^2}{(q - 1)^2}$$

$$E(X^2) = \frac{q(1 - \theta q - q)}{(q - 1)^2} - \frac{2q^2(1 - \theta q - q)}{(q - 1)^3} - \frac{\theta q}{(q - 1)} + \frac{6\theta q^2}{(q - 1)^2} - \frac{6\theta p^3}{(q - 1)^3}$$

$$E(X^3) = \frac{q(1 - \theta q - q)}{(q - 1)^2} - \frac{6q^2(1 - \theta q - q)}{(q - 1)^3} + \frac{6q^3(1 - \theta q - q)}{(q - 1)^4} - \frac{\theta q}{(q - 1)}$$
$$+ \frac{14\theta q^2}{(q - 1)^2} - \frac{36\theta q^3}{(q - 1)^3} + \frac{24\theta q^4}{(q - 1)^4}$$

$$E(X^4) = \frac{q(1 - \theta q - q)}{(q - 1)^2} - \frac{14q^2(1 - \theta q - q)}{(q - 1)^3} + \frac{36q^3(1 - \theta q - q)}{(q - 1)^4} - \frac{24q^4(q - \theta q + 1)}{(q - 1)^5}$$
$$- \frac{\theta q}{(q - 1)} + \frac{30\theta q^2}{(q - 1)^2} - \frac{150\theta q^3}{(q - 1)^3} + \frac{240\theta q^4}{(q - 1)^4} - \frac{120\theta q^5}{(q - 1)^5}$$

The estimator of θ by using moments method can be found as

$$\frac{q(1 - \theta q - q) - \theta q(q - 1) + 2\theta q^2}{(q - 1)^2} = \mu_1 \tag{2.8}$$
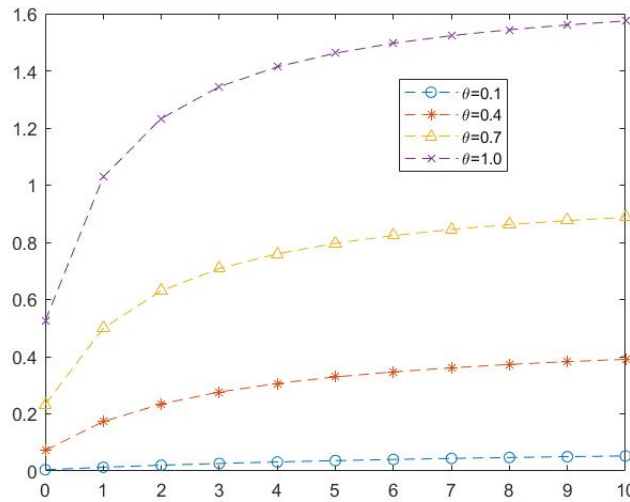
Fig. 2. Possible shapes of hazard rate function of the DLBE distribution

where $q = exp(-\theta)$ and $\mu_1 = \bar{X}$. The method of moments estimator of θ is the root of this equation. For small θ, the approximation of $q = 1 - \theta$ can be used as

$$-\theta^2(\mu_1 + 3) + 2\theta + 1 = 0 \tag{2.9}$$

for better approximation the better approximation $p = 1 - \theta + \frac{\theta^2}{2}$ can be used.

In order to obtain the ML estimator of the unknown parameter, the log-likelihood function of θ is

$$logL(\theta) = -\theta \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} log\left(\theta X_i(1 - exp(-\theta)) + 1 - \theta exp(-\theta) - exp(-\theta)\right) \tag{2.10}$$

Taking the derivative of the log-likelihood function with respect to the unknown parameter θ and equating it to the zero, we obtain the following equation

$$\frac{\partial \ln L}{\partial \theta} = -\sum_{i=1}^{n} X_i + \sum_{i=1}^{n} \frac{X_i + exp(-\theta)(\theta X_i - X_i + \theta)}{\theta X_i(1 - exp(-\theta)) + 1 - \theta exp(-\theta) - exp(-\theta)} \tag{2.11}$$

The solution is called maximum likelihood estimator (MLE) of θ and has no explicit solutions. Iterative methods can be used for solving the equation.

3 Discrete Area-Biased Exponential Distribution

The probability mass function of the discrete area-biased exponential (DABE) distribution can be obtained by using Equation (1.1) and Equation (1.7) as follows:

$$P(X = x) = q^x \left(\frac{\theta^2}{2}x^2(1 - q) + \theta x(1 - \theta q - q) + (1 - \frac{\theta^2}{2}q - \theta q - q)\right), x = 0, 1, 2, ... \tag{3.1}$$

where $q = exp(-\theta)$. The corresponding cumulative distribution function and survival function of the DABE distribution are

$$F(x) = 1 - q^{x+1}\Big(\frac{\theta^2}{2}x^2 + \theta^2 x + \theta x + \frac{\theta^2}{2} + \theta + 1\Big), x = 0, 1, 2, ... \qquad (3.2)$$

and

$$S(x) = q^{x+1}\Big(\frac{\theta^2}{2}x^2 + \theta^2 x + \theta x + \frac{\theta^2}{2} + \theta + 1\Big), x = 0, 1, 2, ... \qquad (3.3)$$

respectively.

Fig. 3 shows some possible shapes of Equation (3.1). It can be seen that the probability mass function is always unimodal.
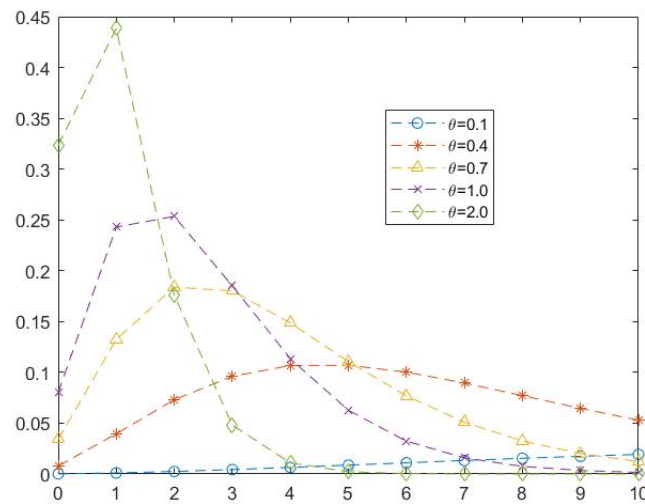

Fig. 3. Possible shapes of mass function of DABE distribution

Fig. 4 shows some possible shapes of hazard rate function $h(x) = f(x)/S(x)$ for different θ values. It can be easily seen that the failure rate is increasing with respect to x and θ values.

As a result, the DABE distribution is unimodal, since the probability function satisfies the log-concave inequality $\big(f^2(x) > f(x+1)f(x-1)\big)$, see Keilson and Gerber, [21]. Additionally, the DABE distribution is unimodal and has a discrete increasing failure rate.

The moment generating function can be found by using probability generating function $E(t^X)$ as:

$$M_x(t) = \frac{\theta^2}{2}(1-q)\frac{e^t q(1+qe^t)}{(1-qe^t)^3} + \theta(1-\theta q - q)\frac{qe^t}{(1-qe^t)^2} + (1 - \frac{\theta^2}{2}q - \theta q - q)\frac{1}{1-qe^t} \qquad (3.4)$$

Consequently, the first four moment of $DLBE$ distribution can be found by using the derivatives of moment generating function obtained in Equation (3.4).
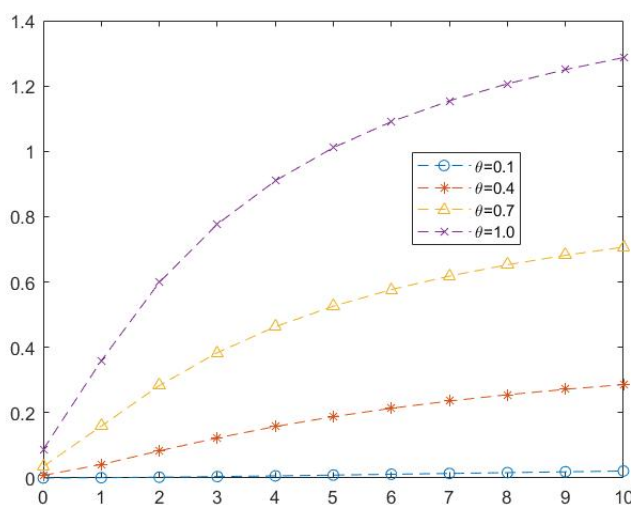
Fig. 4. Possible shapes of hazard rate function of DABE distribution

$$E(X) = \frac{\theta^2 q}{2(q-1)^2} - \frac{q(1-\theta^2/q-q\theta-q)}{(q-1)^2} - \frac{\theta q(1-\theta q-q)}{(q-1)^2} - \frac{3\theta^2 q(q+1)}{2(q-1)^3}$$
$$+ \frac{2\theta^2(1-\theta q-q)}{(q-1)^3}$$

$$E(X^2) = \frac{2q^2(1-\theta^2/q-q\theta-q)}{(q-1)^3} - \frac{3\theta^2 q^2}{(q-1)^3} + \frac{\theta^2 q}{2(q-1)^2} - \frac{q(1-\theta^2/q-q\theta-q)}{(q-1)^2}$$
$$- \frac{q(1-\theta q-q)}{(q-1)^2} - \frac{3\theta^2 q(q+1)}{2(q-1)^3} + \frac{6\theta^2 q^2(1-\theta q-q)}{(q-1)^3} - \frac{6\theta^2 q^3(1-\theta q-q)}{(q-1)^4} + \frac{6\theta^2 q^2(q+1)}{(q-1)^4}$$

$$E(X^3) = \frac{180\theta^2 q^3}{(q-1)^4} - \frac{9\theta^2 q^2}{(q-1)^3} + \frac{6q^2(1-\theta^2/q-q\theta-q)}{(q-1)^3} - \frac{6q^3(1-\theta^2/q-q\theta-q)}{(q-1)^4}$$
$$+ \frac{\theta^2 q}{2(q-1)^2} - \frac{q(1-\theta^2/q-q\theta-q)}{(q-1)^2} - \frac{\theta q(1-\theta q-q)}{(q-1)^2} - \frac{3\theta^2 q(q+1)}{2(q-1)^3} + \frac{14\theta q^2(1-\theta q-q)}{(q-1)^3}$$
$$- \frac{36\theta q^3(1-\theta q-q)}{(q-1)^4} + \frac{24\theta q^4(1-\theta q-q)}{(q-1)^5} + \frac{18\theta^2 q^2(q+1)}{(q-1)^4} - \frac{30\theta^2 q^3(q+1)}{(q-1)^5}$$

$$E(X^4) = \frac{108\theta^2 q^3}{(q-1)^4} - \frac{21\theta^2 q^2}{(q-1)^3} - \frac{120\theta^2 q^4}{(q-1)^5} + \frac{14q^2(1-\theta^2/q-q\theta-q)}{(q-1)^3}$$
$$- \frac{36q^3(1-\theta^2/q-q\theta-q)}{(q-1)^4} + \frac{24q^4(1-\theta^2/q-q\theta-q)}{(q-1)^5} + \frac{\theta^2 q}{2(q-1)^2} - \frac{q(1-\theta q-q)}{(q-1)^2}$$
$$- \frac{q(1-\theta^2/q-q\theta-q)}{(q-1)^2} - \frac{30\theta^2 q(q+1)}{2(q-1)^3} + \frac{30\theta q^2(1-\theta q-q)}{(q-1)^3} - \frac{150\theta q^3(1-\theta q-q)}{(q-1)^4}$$
$$+ \frac{240\theta q^4(1-\theta q-q)}{(q-1)^5} - \frac{120\theta q^5(1-\theta q-q)}{(q-1)^6} + \frac{42\theta^2 q^2(q+1)}{(q-1)^4} - \frac{180\theta^2 q^3(q+1)}{(q-1)^5}$$
$$+ \frac{180\theta^2 q^4(q+1)}{(q-1)^6}$$

The estimator of θ by using moments method can be found as

$$E(X) = \frac{\theta^2 q}{2(q-1)^2} - \frac{q(1-\theta^2/q-q\theta-q)}{(q-1)^2} - \frac{q(1-\theta q-q)}{(q-1)^2} - \frac{3\theta^2 q(q+1)}{2(q-1)^3} = \mu_1 \tag{3.5}$$

where $q = exp(-\theta)$ and $\mu_1 = \bar{X}$. The method of moments estimator of θ is the root of this equation. For small θ, the approximation of $q = 1 - \theta$ can be used. For better approximation the better approximation $p = 1 - \theta + \frac{\theta^2}{2}$ can be used and the solutions are the roots of the following function

In order to obtain the ML estimator of the unknown parameter, the log-likelihood function of θ is

$$logL(\theta) = -\theta \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} log\left(\frac{\theta^2}{2}X_i^2(1 - exp(-\theta))\right. \tag{3.6}$$

$$\left. +\theta X_i(1 - \theta exp(-\theta) - exp(-\theta)) + (1 - \frac{\theta^2}{2}exp(-\theta) - \theta exp(-\theta) - exp(-\theta))\right)$$

Taking the derivative of the log-likelihood function with respect to the unknown parameter θ and equating it to the zero, we obtain the following equation

$$\tag{3.7}$$

$$-\sum_{i=1}^{n} X_i + \sum_{i=1}^{n} \frac{\theta^2 e^{-\theta} - X_i e^{-\theta} + \theta e^{-\theta} - 1 - \theta X_i(e^{-\theta} - 1)}{2\left(\frac{\theta^2}{2}X_i(1 - e^{-\theta}) + \theta X_i(1 - \theta e^{-\theta} - e^{-\theta}) + (1 - \frac{\theta^2}{2}e^{-\theta} - \theta e^{-\theta} - e^{-\theta})\right)}$$

The solution is called MLE of θ and has no explicit solutions. Iterative methods can be used for solving the equation.

4 Simulation Study

In this section, we perform a simulation study for the performance of the maximum-likelihood estimators. First, we generate 10,000 samples of size n and we compute the ML estimates for the 10,000 samples, say $\hat{\theta}$ for $i = 1, 2, ...$. Then, we compute the estimates, biases, standard errors (SE), mean-squared errors (MSE) and confidence intervals for the DLBE distribution as

1. $E(\hat{\theta}) = \frac{1}{n}\sum_{i=1}^{n} \hat{\theta}_i = \bar{\theta}$

2. $Bias(\hat{\theta}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\theta}_i - \theta)$

3. $E(SE(\hat{\theta})) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{-\partial^2 \ln L}{\partial \hat{\theta}_i^2}\right)}$

4. $MSE(\hat{\theta}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\theta}_i - \theta)^2$

5. $CI_{95\%} = \hat{\theta} \pm 1.96\, SE(\hat{\theta})$

We repeat these steps for $n = 10, 20$ and 50 with $\theta = 0.1, 0.4, 1.0$ and 2.0. Table 1 shows the result of the simulation.

Table 1. Simulation results for MLE

True Parameter	Estimates	Sample Size		
		10	25	50
	$E(\hat{\theta})$	0.087	0.089	0.093
	$Bias(\hat{\theta})$	0.006	0.002	0.001
$\theta = 0.1$	$E(SE(\hat{\theta}))$	0.006	0.002	0.001
	$MSE(\hat{\theta})$	0.005	0.001	0.001
	$\%p\ in\ CI_{95\%}$	%91.8	%92.4	%93.1
	$E(\hat{\theta})$	0.367	0.377	0.384
	$Bias(\hat{\theta})$	0.003	0.001	0.001
$\theta = 0.4$	$E(SE(\hat{\theta}))$	0.004	0.003	0.003
	$MSE(\hat{\theta})$	0.002	0.001	0.001
	$\%p\ in\ CI_{95\%}$	%92.7	%93.4	%94.2
	$E(\hat{\theta})$	0.973	0.989	1.005
	$Bias(\hat{\theta})$	0.002	0.002	0.001
$\theta = 1.0$	$E(SE(\hat{\theta}))$	0.004	0.003	0.003
	$MSE(\hat{\theta})$	0.002	0.001	0.001
	$\%p\ in\ CI_{95\%}$	%92.8	%93.5	%94.4
	$E(\hat{\theta})$	1.975	1.989	1.996
	$Bias(\hat{\theta})$	0.003	0.001	0.001
$\theta = 2.0$	$E(SE(\hat{\theta}))$	0.004	0.002	0.002
	$MSE(\hat{\theta})$	0.002	0.001	0.001
	$\%p\ in\ CI_{95\%}$	%93.0	%93.8	%94.5

From the Table 1, it can be seen that as the sample size increases, the bias, the mean square error, and the standard error of the estimates decreases. Further, the estimators satisfy the normality from the confidence intervals. The same procedures is repeated for the DABE distribution. The results and the interpretation of these results are approximately same.

5 Application to Real Data

In this section, we use some real-life data that has been applied to other discrete distribution. The first data set given in Table 2 consists of survival times in days of 72 guinea pigs given by Bjerkedal [22]. The data have been analyzed by many authors, recently Bakouch et al. [8]used this data for discrete Lindley distribution. In this paper, the data were fitted to the five models (discrete Lindley, geometric, Poisson, discrete Weibull and discrete gamma) and shown that the best fit is discrete Lindley distribution.

Table 2. Data Set 1

12	15	22	24	24	32	32	33	34	38	38	43
44	48	52	53	54	54	55	56	57	58	58	59
60	60	60	60	61	62	63	65	65	67	68	70
70	72	73	75	76	76	81	83	84	85	87	91
95	96	98	99	109	110	121	127	129	131	143	146
146	175	175	211	233	258	258	263	297	341	341	376

We fit the DLBE distribution to the guinea pig's data. In Table 3, Kolmogrov-Smirnov test statistics, p-values, log-likelihood and AIC values for discrete Lindley and the proposed distribution.

Table 3. Fitted estimates for data set 1

Model	Estimates	p-value	K-S stat	lnL	AIC
Discrete Lindley	$\hat{\theta} = 0.019$	0.0683	0.1507	-252.716	507.44
DLBE	$\hat{\theta} = 0.016$	0.0705	0.1685	-237,747	477.49

As can be seen from Table 3, the guinea pigs data provides better fits than discrete Lindley and of course other four models geometric, Poisson, discrete Weibull and discrete gamma with bigger lnL, p-value and smaller AIC values.

The second data set is about the number of European red mites on apple leaves [23, 24, 25, 26, 27]. Chakraborty [6] proposed discrete gamma distribution and compared with negative binomial and generalized Poisson distribution. The data set is given at Table 4.

Table 4. Data Set 2

European redmites	Frequency
0	70
1	38
2	17
3	10
4	9
5	3
6	2
7	1

We fit the DLBE distribution to the European redmites data. In Table 5, Kolmogrov-Smirnov test statistics, p-values, log-likelihood and AIC values for discrete gamma and the proposed distribution.

Table 5. Fitted estimates for data set 2

Model	Estimates	p-value	K-S stat	lnL	AIC
Discrete gamma	$\hat{\theta} = 1.583, \hat{k} = 1.007$	0.7160	0.1489	-222.44	448.88
DLBE	$\hat{\theta} = 0.65$	0.7528	0.1447	-183.89	365.77

As can be seen from Table 5, the European redmites data provides better fits than discrete gamma and of course other two models generalized Poisson and negative binomial with bigger lnL, p-value and smaller AIC values.

The last data set given in Table 6 consists of remission times in weeks for 20 leukaemia patients randomly assigned to a certain treatment taken from Lawless [28]. Bakouch et al. [8] used this data for discrete Lindley distribution. In this paper, the data were fitted to the five models (discrete Lindley, geometric, Poisson, discrete Weibull and discrete gamma) and shown that the best fit is discrete Lindley distribution.

We fit the DABE distribution to the leukaemia patients data. In Table 7, Kolmogrov-Smirnov test statistics, p-values, log-likelihood and AIC values for discrete Lindley and the proposed distribution.

Table 6. Data Set 3

1	3	3	6	7	7	10	12	14	15	18	19
22	26	28	29	34	40	48	49				

Table 7. Fitted estimates for data set 3

Model	Estimates	p-value	K-S stat	lnL	AIC
Discrete Lindley	$\hat{\theta} = 0.10$	0.0541	0.2909	-49.06	96.12
DABE	$\hat{\theta} = 0.08$	0.126	0.1786	-30.63	59.26

As can be seen from Table 7 the leukaemia patients data provides better fits than discrete Lindley and of course other four models geometric, Poisson, discrete Weibull and discrete gamma with bigger lnL, p-value and smaller AIC values.

6   Conclusion

In recent years, it has been seen that, traditional statistical distributions have limited application areas. For this reason, some authors have proposed some generalization methods of these distributions or some other new distributions. In this paper, we propose a discrete length and discrete area biased exponential distributions. We obtain some statistical and mathematical properties of these distributions. We also get likelihood equations of the corresponding proposed distributions. In real life data analysis section, we fit some data to the discrete length and area exponential distribution and observe that these proposed distributions give more reliable and better solutions than the alternative models.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Chouia S, Zeghdoudi H. The XLindley distribution: Properties and Application. Journal of Statistical Theory and Applications. 2021;20(2):318-327.

[2] Nakagawa T, Osaki S. Discrete Weibull distribution. IEEE Trans. Reliab. 1975;24:300–301.

[3] Stein WE, Dattero R. A new discrete Weibull distribution. IEEE Trans. Reliab. 1984;33:196–197.

[4] Roy D. The discrete normal distribution. Commun. Statist. Theor. Meth. 2003;32(10):1871–1883.

[5] Krishna H, Pundir PS. Discrete Burr and discrete Pareto distributions. Statist. Methodol. 2009;6:177–188.

[6] Chakraborty S, Chakravarty D. Discrete gamma distributions: Properties and Parameter Estimations. Communications in Statistics - Theory and Methods. 2012;41(18):3301-3324.

[7] Jazi MA, Lai CD, Alamatsaz MH. A discrete inverse Weibull distribution and estimation of its parameters. Statist. Methodol. 2009;7:121–132.

[8] Bakouch HS, Jazi MA, Nadarajah S. A new discrete distribution. Statistics A Journal of Theoretical and Applied Statistics. 2014;48(1).

[9] Nekoukhou V, Alamatsaz MH, Bidram H. Discrete generalized exponential distribution of a second type. Statistics. 2013;47:876–887.

[10] Roy D. Discretization of continuous distributions with an application to stress–strength reliability. Calcutta Statist. Assoc. Bull. 2002;52:297–313.

[11] Lin TM, Guillén M. The rising hazards of party incumbency: A discrete renewal analysis. Political Anal. 1998;7:31–57.

[12] Haas M. Improved duration-based backtesting of value-at-risk. J. Risk. 2005;8:17–38.

[13] Wang LC, Yang Y, Zou Y, Yu YL, Kang R. Minimal availability variation design of repairable system under discrete Weibull distribution. Control Theory Appl. 2010;27:575–581.

[14] Gupta RD, Kundu D. Generalized exponential distributions. Austral. NZ J. Statist. 1999;41(2):173–188.

[15] Gupta RD, Kundu D. Exponentiated exponential family: An alternative to gamma and Weibull distributions. Biom. J. 2001;43:117–130.

[16] Iqbal Z, Hasnain SA, Salman M, Ahmad M, Hamedani D. Generalized exponentiated moment exponential distribution. Pakistan J. Stat. Krishna; 2014.

[17] Abu-Youssef S, Mohammed B, Sief M. An extended exponentiated exponential distribution and its properties. Int. J. Comput. Appl. 2015;121.

[18] Pogány TK, Saboor A, Provost S. The Marshall-Olkin exponential Weibull distribution. Hacettepe J. Math. Stat. 2015;44:1579–1594.

[19] Ristić MM, Kundu D. Marshall-Olkin generalized exponential distribution. Metron 2015;73(3):317–333.

[20] Hasnain SA, Iqbal Z, Ahmad M. On exponentiated moment exponential distribution. Pak. J. Statist. 2015;31(2):267–280.

[21] Keilson J, Gerber H. Some results for discrete unimodality. J. Am. Statist. Assoc. 1971;66:386–389.

[22] Bjerkedal T. Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. Am. J. Hyg. 1960;72:130–148.

[23] Bliss CI. Fitting the negative binomial distribution to biological data. Biometrics. 1953;(9)177–200.

[24] Ong SH, Lee PA. On a generalized non-central negative binomial distribution. Commun. Statist. Theor. Meth. 1986;15(3):1065–1079.

[25] Gupta RC, Ong SH. A new generalization of the negative binomial distribution. Computat. Statist. Data Anal. 2004;45:287–300.

[26] Hand DJ, Daly F, Lunn AD, McConway KJ, Ostrowski EO. A Hand Book of Small Data Sets. London: Chapman and Hall; 1994.

[27] Chakraborty S. On some distributional properties of the family of weighted generalized Poisson distribution. Commun. Statist. Theor. Meth. 2010;39:2767–2788.

[28] Lawless JF. Statistical Models and Methods for Lifetime Data, 2nd ed. John Wiley and Sons, New York; 2003.