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THE IMPLICATIONS OF N=2 SUPERGRAVITY COSMOLOGYON THE TOPOLOGY OF THE CALABI-YAU MANIFOLD

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ABSTRACT

When N = 1 $D = 11$ supergravity is compactified on CY threefold to N = 2 $D = 5$ supergravity, the action of the last is given in terms of the geometry of the CY manifold space, namely, in terms of the hypermultiplets. There are $(z^i : i = 1,\ldots,h_{2,1})$ complex structure moduli in the moduli space of the CY manifold which's a special $K\ddot{a}hler$ manifold with a metric $G_{i\bar{j}}$. We solve the field equations of the complex structure moduli with the solution of the Einstein field equations to the moduli velocity norm $G_{i\bar{j}}\, \dot{z}^i \dot{z}^{\bar{j}}$ in the case of a 3- brane filled with radiation, dust, and energy embedded in the bulk of $D = 5$ supergravity. We get the time dependence of the moduli and the metric. Then we can further deduce the geometry of the moduli space by getting the $K\ddot{a}hler$ potential that directly relates to the volume of the CY manifold.

Keywords: Supergravity; Cosmology; General relativity; Extra dimensions; Calabi-Yau manifold.

1. INTRODUCTION

The compactification of string theory over Calabi-Yau manifolds yields two sets of parameters [1, 2]. The parameters correspond to the structure of Calabi-Yau manifold M as a complex manifold and the deformation of the $K\ddot{a}hler$ metric of the complex structure space. And parameters corresponding to the deformation of M as a complex $K\ddot{a}hler$ manifold M_K Calabi-Yau 3-folds admit H^3 homolgy group that can be Hodge decomposed as:

$$H^3 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}. \qquad (1)$$

So CY 3-folds have a single (3,0) cohomology form, where the hodge number $h_{(3,0)} = \dim(H^{3,0}) = 1$. We will call the holomorphic volume form as Ω, (2,1) forms related to M_C, with a Hodge number $h_{2,1}$ determines the dimensions of M_C, and (1,2) forms related to M_K, with a Hodge number $h_{1,2}$ determines the dimensions of M_K. The deformation of M can be done by either the deformation of M_C or the deformation of the $K\ddot{a}hler$ form K of M_K or both. The $K\ddot{a}hler$ form is defined by:

$$K = i\, g_{m\bar{n}}\ d\omega^m \wedge d\omega^{\bar{n}}. \qquad (2)$$

The holomorphic coordinates $i, \bar{J} = 1, \dots, m$, where $2\,m$ is the dimension of the manifold. The CY metric is defined by

$$g_{m\bar{n}} = \partial_m \partial_{\bar{n}}\, \kappa, \qquad (3)$$

where κ is the K$\ddot{a}$hler potential. This paper is devoted to explore the space of the complex structure moduli M_C that is described by the (2,1) forms

$$\chi_{i|mn\bar{p}} = -\Omega^{\bar{r}}_{mn} \left(\frac{\partial g_{\bar{p}\bar{r}}}{\partial z^i} \right), \qquad (4)$$

where (z^i : i= 1,...,$h_{2,1}$) are the parameters or the moduli of the complex structure space. Each "i" defines a (2,1) cohomology class. It's important here to declare that z^i can be treated as complex coordinates that define a $K\ddot{a}hler$ metric $G_{i\bar{J}}$ on M_C as follows:

$$V_{CY}\ G_{i\bar{j}}(\delta z^i)(\delta \bar{z}^{\bar{j}}) = \frac{1}{4} \int_{\mathcal{M}} g^{m\bar{n}}\ g^{r\bar{p}}\ (\delta g_{mr})\ (\delta g_{\bar{n}\bar{p}}) \qquad (5)$$

where V_{CY} is the CY volume. $G_{i\bar{J}}$ is related to the K$\ddot{a}$hler potential by

$$G_{i\bar{J}} = \partial_i\ \partial_{\bar{J}}\ \kappa, \qquad (6)$$

that leads to a relation between κ and the volume form

$$\int_M \Omega \wedge \bar{\Omega} = -ie^{-\kappa} \qquad (7)$$

gives that the K$\ddot{a}$hler potential is related to the volume of CY manifold simply by [3, 4]

$$\mathrm{Vol}\ (\mathrm{M}) = e^{-\kappa}\,. \qquad (8)$$

We will consider here the Hodge number $h_{2,1}$ = 1, i.e., we have only one moduli z, a single K$\ddot{a}$hler metric component G, and the dimension of M_C is unity. In this work we aim to find the time dependence of the scalar quantity $G_{i\bar{J}}\, z^i z^{\bar{J}}$ and then to find:

- The variation of the moduli z and the Kähler metric G with time.
- The time dependence of the Kähler potential κ and the Vol (M).

Our study is based on D = 5 N = 2 supergravity where the universe is modeled as a 3-brane embedded in a 5-dimensional bulk. Previously [9] we have found that the moduli's velocity norm $G_{i\bar{J}}\, z^i z^{\bar{J}}$ correlates to the scale factors of the brane universe or the bulk and significantly corresponds to our own universe cosmological time evolution. We have studied a 3-brane filled by radiation as our very early universe and a brane filled by dust, where the Friedmann-like equations have been numerically solved for a wide range of the scale factors' initial conditions. In all different cases $G_{i\bar{J}}\, z^i z^{\bar{J}}$ manifested itself as an agent starts with very large values causes an early epoch of rapid expansion (inflation) then it decays fastly to asymptotic values. Here, we will extend that study and add to Einstein's equations a cosmological constant term. We will solve the field equations in the case of a 3-brane filled by radiation, dust and energy (cosmological constant Λ), while we consider the bulk's cosmological constant $\tilde{\Lambda}$ vanishes. Then we will use these results to explore the non-trivial topology of the Calabi-Yau manifold. We use the system of units M_p= 1 (M_p= 2.4 x 10^{18} GeV = $(8\,\pi\, G)^{-1/2}$).

It's worthy to mention that there are many studies about the geometry of the moduli spaces for a Calabi-Yau manifold like [5, 6, 7, 8], where at [5] for instance CY 3-fold was considered as a quintic threefold in the P^4

projection space with $h_{2,1}$ = 101. However, we don't need here to make this assumption.

So this paper is organized as follows: in section 1 we introduce the D = 5 N = 2 supergravity as the dimensional reduction of eleven-dimensional supergravity theory over a Calabi-Yau 3-fold M . Then we will solve the field equations. In section 2 we will simplify the moduli field equations and the Kähler metric equation and show how they can be solved using the solution of the moduli's velocity norm $G_{i\bar{j}} z^i z^{\bar{j}}$. In the same section, we will introduce the time dependence of the Kähler metric, the Kähler potential, and the volume of CY manifold.

1. D=5 N =2 supergravity and its solution

The five-dimensional N = 2 supergravity theory contains two sets of matter fields; the vector multiplets, which we set to zero, and our main interest: the *hypermultiplets*. These are composed of the *universal hypermultiplet* $(\phi, \sigma, \zeta^0, \tilde{\zeta}_0)$; where ϕ is the universal axion, and the dilaton is proportional to the volume of the underlying Calabi-Yau manifold M. The remaining hypermultiplet scalars are $z^i, z^{\bar{\imath}}, \zeta^i, \tilde{\zeta}_i : i = 1, \ldots, h_{2,1}$, where the z's are the complex structure moduli of M, and $h_{2,1}$ is the Hodge number determining the dimensions of the manifold M_C of the Calabi- Yau's complex structure moduli[1]. The fields $(\zeta^i, \tilde{\zeta}_i : I = 0, \ldots, h_{2,1})$ are the axions, which define a symplectic vector space (see [4] for a review and more references). The axions are defined as components of the symplectic vector:

$$|\Xi\rangle = \begin{pmatrix} \zeta^I \\ \tilde{\zeta}_I \end{pmatrix} \tag{9}$$

such that the symplectic scalar product is defined by, for example,

$$\langle \Xi | \tilde{\Xi} \rangle = \zeta^I \tilde{\zeta}_I - \tilde{\zeta}_I \zeta^I. \tag{10}$$

A transformation in symplectic space can be defined by

$$\langle d\Xi | \underset{\wedge}{\boldsymbol{\Lambda}} | \star d\Xi \rangle = 2 \langle d\Xi \mid V \rangle \underset{\wedge}{} \langle \bar{V} \mid \star d\Xi \rangle + 2G^{i\bar{j}} \langle d\Xi \mid U_{\bar{j}} \rangle \underset{\wedge}{} \langle U_i \mid \star d\Xi \rangle - i \langle d\Xi \underset{\wedge}{\mid} \star d\Xi \rangle \tag{11}$$

where d is the spacetime exterior derivative, $\star$ is the five dimensional Hodge duality operator, and $G_{i\bar{j}}$ is a special Kähler metric on M_C. The symplectic basis vectors $|V\rangle, |U_i\rangle$, and their complex conjugates are defined by

$$|V\rangle = e^{\frac{\mathcal{K}}{2}} \begin{pmatrix} Z^I \\ F_I \end{pmatrix} \qquad |\bar{V}\rangle = e^{\frac{\mathcal{K}}{2}} \begin{pmatrix} \bar{Z}^I \\ \bar{F}_I \end{pmatrix} \tag{12}$$

where κ is the Kähler potential on M_C, (Z, F) are the periods of the Calabi-Yau's holomorphic volume form, and

$$\begin{aligned} |U_i\rangle &= |\nabla_i V\rangle = \left| \left[\partial_i + \frac{1}{2} (\partial_i \mathcal{K}) \right] V \right\rangle \\ |U_{\bar{i}}\rangle &= |\nabla_{\bar{i}} \bar{V}\rangle = \left| \left[\partial_{\bar{i}} + \frac{1}{2} (\partial_{\bar{i}} \mathcal{K}) \right] \bar{V} \right. \end{aligned} \tag{13}$$

where the derivatives are with respect to the moduli ($z^i, z^{\bar{\imath}}$). In this language, the bosonic part of the action is given by:

$$S_5 = \int_5 [\, R \star 1 - \tfrac{1}{2} d\sigma \wedge\star d\sigma - G_{i\bar{j}} dz^i \wedge\star dz^{\bar{j}} + e^{\sigma} \langle d\,\Xi | \boldsymbol{\Lambda} | \star d\Xi \rangle - \tfrac{1}{2} e^{2\sigma} [d\phi + \langle \Xi | d\Xi \rangle] \wedge\star [d\phi + \langle \Xi | d\Xi \rangle] \,]. \tag{14}$$

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[1]A 'bar' over an index denotes complex conjugation

The usual $\delta S = 0$ gives the following field equations for the hypermultiplets scalar fields:

$$(\Delta\sigma)\star\mathbf{1} + e^{\sigma}\left\langle d\Xi\right| \underset{\wedge}{\mathbf{\Lambda}} \left|\star d\Xi\right\rangle - e^{2\sigma}\left[d\phi + \left\langle \Xi \mid d\Xi\right\rangle\right]\wedge\star\left[d\phi + \left\langle \Xi \mid d\Xi\right\rangle\right] = 0 \tag{15}$$

$$\left(\Delta z^{i}\right)\star\mathbf{1} + \Gamma^{i}_{jk}dz^{j}\wedge\star dz^{k} + \frac{1}{2}e^{\sigma}G^{i\bar{j}}\partial_{\bar{j}}\left\langle d\Xi\right| \underset{\wedge}{\mathbf{\Lambda}} \left|\star d\Xi\right\rangle = 0$$

$$\left(\Delta z^{\bar{i}}\right)\star\mathbf{1} + \Gamma^{\bar{i}}_{\bar{j}\bar{k}}dz^{\bar{j}}\wedge\star dz^{\bar{k}} + \frac{1}{2}e^{\sigma}G^{\bar{i}j}\partial_{j}\left\langle d\Xi\right| \underset{\wedge}{\mathbf{\Lambda}} \left|\star d\Xi\right\rangle = 0 \tag{16}$$

$$d^{\dagger}\left\{e^{\sigma}\left|\mathbf{\Lambda} d\Xi\right\rangle - e^{2\sigma}\left[d\phi + \left\langle \Xi \mid d\Xi\right\rangle\right]\left|\Xi\right\rangle\right\} = 0 \tag{17}$$

$$d^{\dagger}\left[e^{2\sigma}d\phi + e^{2\sigma}\left\langle \Xi \mid d\Xi\right\rangle\right] = 0 \tag{18}$$

where $d^{\dagger}$ is the D = 5 adjoint exterior derivative, Δ is the Laplace-de Rahm operator and Γ^{i}_{jk} is a connection on M_C. The full action is symmetric under the following SUSY transformations:

$$\begin{aligned}
\delta_{\epsilon}\psi^{1} &= D\epsilon_{1} + \frac{1}{4}\left\{ie^{\sigma}\left[d\phi + \left\langle \Xi \mid d\Xi\right\rangle\right] \quad Y\right\}\epsilon_{1} \quad e^{\frac{\sigma}{2}}\left\langle \bar{V} \mid d\Xi\right\rangle\epsilon_{2}\\
\delta_{\epsilon}\psi^{2} &= D\epsilon_{2} \quad \frac{1}{4}\left\{ie^{\sigma}\left[d\phi + \left\langle \Xi \mid d\Xi\right\rangle\right] \quad Y\right\}\epsilon_{2} + e^{\frac{\sigma}{2}}\left\langle V \mid d\Xi\right\rangle\epsilon_{1}\\
\delta_{\epsilon}\xi^{0}_{1} &= e^{\frac{\sigma}{2}}\left\langle V \mid \partial_{\mu}\Xi\right\rangle\Gamma^{\mu}\epsilon_{1} \quad \left\{\frac{1}{2}\left(\partial_{\mu}\sigma\right) \quad \frac{i}{2}e^{\sigma}\left[\left(\partial_{\mu}\phi\right) + \left\langle \Xi \mid \partial_{\mu}\Xi\right\rangle\right]\right\}\Gamma^{\mu}\epsilon_{2}
\end{aligned} \tag{19}$$

$$\delta_{\epsilon}\xi^{0}_{2} = e^{\frac{\sigma}{2}}\left\langle \bar{V} \mid \partial_{\mu}\Xi\right\rangle\Gamma^{\mu}\epsilon_{2} + \left\{\frac{1}{2}\left(\partial_{\mu}\sigma\right) + \frac{i}{2}e^{\sigma}\left[\left(\partial_{\mu}\phi\right) + \left\langle \Xi \mid \partial_{\mu}\Xi\right\rangle\right]\right\}\Gamma^{\mu}\epsilon_{1} \tag{20}$$

and:

$$\begin{aligned}
\delta_{\epsilon}\xi^{\hat{i}}_{1} &= e^{\frac{\sigma}{2}}e^{\hat{i}j}\left\langle U_{j} \mid \partial_{\mu}\Xi\right\rangle\Gamma^{\mu}\epsilon_{1} \quad e^{\hat{i}}{}_{\bar{j}}\left(\partial_{\mu}z^{\bar{j}}\right)\Gamma^{\mu}\epsilon_{2}\\
\delta_{\epsilon}\xi^{\hat{i}}_{2} &= e^{\frac{\sigma}{2}}e^{\hat{\bar{i}}\bar{j}}\left\langle U_{\bar{j}} \mid \partial_{\mu}\Xi\right\rangle\Gamma^{\mu}\epsilon_{2} + e^{\hat{i}}{}_{j}\left(\partial_{\mu}z^{j}\right)\Gamma^{\mu}\epsilon_{1}
\end{aligned} \tag{21}$$

where (ψ^1, ψ^2) are the two gravitini and (ζ^I_1, ζ^I_2) are the hyperini. The quantity Y is defined by:

$$Y = \frac{\bar{Z}^{I}N_{IJ}dZ^{J} \quad Z^{I}N_{IJ}d\bar{Z}^{J}}{\bar{Z}^{I}N_{IJ}Z^{J}} \tag{22}$$

where $N_{IJ} = Im\,(\partial_I F_I)$ The e's are the beins of the special Kähler metric $G_{i\bar{J}}$, the ε's are the five-dimensional N = 2 SUSY spinors and the Γ 's are the usual Dirac matrices. The covariant derivative D is defined by the usual , where the ω's are the spin connections and the hatted indices are frame indices in flat tangent space. Finally, the stress tensor is:

$$\begin{aligned}
T_{\mu\nu} = \; & \frac{1}{2}\left(\partial_{\mu}\sigma\right)\left(\partial_{\nu}\sigma\right) + \frac{1}{4}g_{\mu\nu}\left(\partial_{\alpha}\sigma\right)\left(\partial^{\alpha}\sigma\right) + e^{\sigma}\left\langle\partial_{\mu}\Xi\right|\mathbf{\Lambda}\left|\partial_{\nu}\Xi\right\rangle - \frac{1}{2}e^{\sigma}g_{\mu\nu}\left\langle\partial_{\alpha}\Xi\right|\mathbf{\Lambda}\left|\partial^{\alpha}\Xi\right\rangle\\
& \frac{1}{2}e^{2\sigma}\left[\left(\partial_{\mu}\phi\right) + \left\langle \Xi \mid \partial_{\mu}\Xi\right\rangle\right]\left[\left(\partial_{\nu}\phi\right) + \left\langle \Xi \mid \partial_{\nu}\Xi\right\rangle\right] + \frac{1}{4}e^{2\sigma}g_{\mu\nu}\left[\left(\partial_{\alpha}\phi\right) + \left\langle \Xi \mid \partial_{\alpha}\Xi\right\rangle\right]\left[\left(\partial^{\alpha}\phi\right) + \left\langle \Xi \mid \partial^{\alpha}\Xi\right\rangle\right]\\
& G_{i\bar{j}}\left(\partial_{\mu}z^{i}\right)\left(\partial_{\nu}z^{\bar{j}}\right) + \frac{1}{2}g_{\mu\nu}G_{i\bar{j}}\left(\partial_{\alpha}z^{i}\right)\left(\partial^{\alpha}z^{\bar{j}}\right).
\end{aligned} \tag{23}$$

As our main interest is bosonic configurations that0 preserve *some* supersymmetry, the stress tensor can be simplified by considering the vanishing of the supersymmetric variations (20, 21); satisfying the BPS condition on the brane. This gives

$$T_{\mu\nu} = G_{i\bar{j}} \left(\partial_\mu z^i \right)\left(\partial_\nu z^{\bar{j}} \right) - \frac{1}{2} g_{\mu\nu} G_{i\bar{j}} \left(\partial_\alpha z^i \right)\left(\partial_\alpha z^{\bar{j}} \right), \quad (24)$$

as was detailed out in [10]. We would like to construct a 3-brane that may be thought of as a flat Robertson-Walker universe embedded in $D = 5$. As such we invoke the metric

$$ds^2 = dt^2 + a^2(t)(dr^2 + r^2\, d\Omega^2) + b^2(t)\, dy^2, \quad (25)$$

where $d\Omega^2 = d\theta^2 + \sin^2(\theta)\, d\phi^2$, $a^2(t)$ is the usual Robertson-Walker scale factor, and $b(t)$ is a possible scale factor for the transverse dimension y (the bulk). The brane is located at $y = 0$ and we ignore all possible y-dependence of the warp factors as well as the hypermultiplet bulk fields; effectively only studying the brane close to its location. In this case, Einstein equations $G_{MN} + \Lambda\, g_{MN} = T_{MN}$ reduce to the Friedmann-like form:

$$3\left[\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{b}}{b}\right)^2\right] = G_{i\bar{j}} z^i z^{\bar{j}} + \rho + \Lambda,$$

$$\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{b}}{b} + 2\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{b}}{b}\right)\right] = G_{i\bar{j}} z^i z^{\bar{j}} + p + \Lambda,$$

$$3\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = G_{i\bar{j}} z^i z^{\bar{j}} - \tilde{\Lambda}. \quad (26)$$

Solving these equations numerically in case the total density equals the dust plus radiation densities $\rho = \rho_r + \rho_m \propto \frac{1}{a^4} + \frac{1}{a^3}$, the total pressure equals to the radiation pressure $p = p_r = \frac{p_r}{3}$, $\Lambda = 1$ (de Sitter space), and $\tilde{\Lambda} =$ yields the brane's, and the bulk's scale factors, and $|G_{i\bar{j}}\, z^i z^{\bar{j}}|$ as functions in time Fig. (1). Using suitable fitting functions we get the solution of the velocity norm of the complex structure moduli given by

$$G_{i\bar{j}}\, z^i z^{\bar{j}}\,(t) \simeq -0.5\,(t + 0.004)^{-0.9}. \quad (27)$$

The brane's scale factor varies with time exponentially $a(t) \sim e^{0.2\, t}$ which means the brane- universe undergoes an inflationary expansion. While the bulk scale factor is given by $b(t) \sim 0.06\, e^{0.87\, t}$.

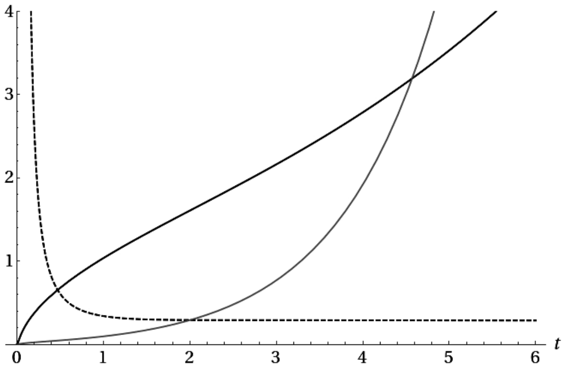

Fig. (1). The scale factor a is represented by the solid curve, b by the grey curve, while $G_{i\bar{j}}\, z^i z^{\bar{j}}$ is shown dashed. $\Lambda = 1$, $\tilde{\Lambda} = 0$, and for initial conditions $a[0] = a'[0] = b[0] = b'[0] = 0$.

As seen the solution shows a correlation between the scale factors of the brane universe and the bulk and the moduli norm $G_{i\bar{j}}\, z^i z^{\bar{j}}$.

2. Calabi-Yau manifold complex structure space

The field equations of the moduli z^i and $z^{\bar{j}}$ (16) can be simplified using the BPS condition [10]:

$$e^{\sigma}\langle \Xi | \underset{\Lambda}{\mathbf{\Lambda}} | \star d\Xi \rangle = \frac{1}{2} d\sigma \wedge \star d\sigma + \frac{1}{2} e^{2\sigma}[d\phi + \langle \Xi |\, d\Xi \rangle] \wedge \star [d\phi + \langle \Xi |\, d\Xi \rangle] + 2G_{i\bar{j}} dz^i \wedge \star dz^{\bar{j}}, \quad (28)$$

so that they can be written as

$$(\Delta z^{i}) \star 1 + \Gamma^{i}_{jk}\, dz^{j} \wedge \star\, dz^{k} + G^{i\bar{j}}\left(\partial_{\bar{j}}\, G_{l\bar{k}}\right) dz^{l} \wedge\star\, dz^{\bar{k}} = 0\,,$$

$$(\Delta z^{\bar{\imath}}\ ) \star 1 + \Gamma^{\bar{\imath}}_{\bar{j}\,\bar{k}}\, dz^{\bar{j}} \wedge \star\, dz^{\bar{k}} + G^{\bar{\imath}j}\left(\partial_{j}\, G_{\bar{l}k}\right) dz^{l} \wedge\star\, dz^{\bar{k}} = 0\,, \tag{29}$$

Dropping the differential forms formulation, we get:

$$\nabla^{2} z^{i} + \Gamma^{i}_{jk}\, \partial_{\mu}\, z^{j}\, \partial^{\mu}\, dz^{k} + G^{i\bar{j}}\left(\partial_{\bar{j}}\, G_{l\bar{k}}\right) \partial_{\mu} dz^{l}\, \partial^{\mu} dz^{\bar{k}} = 0\,,$$

$$\nabla^{2} z^{\bar{\imath}} + \Gamma^{\bar{\imath}}_{\bar{j}\,\bar{k}}\, \partial_{\mu}\, dz^{\bar{j}}\, \partial^{\mu} dz^{\bar{k}} + G^{\bar{\imath}j}\left(\partial_{j}\, G_{\bar{l}k}\right) \partial_{\mu} z^{l}\, \partial^{\mu}\, dz^{\bar{k}} = 0\,. \tag{30}$$

The connections or the Christoffel symbols are related to the metric by [4]:

$$\Gamma^{i}_{jk} = G^{i\bar{p}}\, \partial_{j} G_{k\bar{p}}\,, \quad \Gamma^{\bar{\imath}}_{\bar{j}\,\bar{k}} = G^{p\bar{\imath}}\, \partial_{\bar{j}}\, G_{\bar{k}p}\,, \tag{31}$$

substitute in (30), we get:

$$\backslash\nabla^{2} z^{i} + G^{i\bar{p}}\, \partial_{j} G_{k\bar{p}}\ \partial_{\mu}\, z^{j}\, \partial^{\mu}\, dz^{k} + G^{i\bar{j}}\left(\partial_{\bar{j}}\, G_{l\bar{k}}\right) \partial_{\mu} dz^{l}\, \partial^{\mu} dz^{\bar{k}} = 0\,,$$

$$\nabla^{2} z^{\bar{\imath}} + G^{p\bar{\imath}}\, \partial_{\bar{j}}\, G_{\bar{k}p}\, \partial_{\mu}\, dz^{\bar{j}}\, \partial^{\mu} dz^{\bar{k}} + G^{\bar{\imath}j}\left(\partial_{j}\, G_{\bar{l}k}\right) \partial_{\mu} z^{l}\, \partial^{\mu}\, dz^{\bar{k}} = 0\,. \tag{32}$$

The moduli are independent of the 3-spatial dimensions. And consider the Hodge number $h_{2,1} = 1$, which means we have only one moduli z, its complex conjugate z^{*}, a single Kähler metric component G, and the dimension of the moduli space M_{C} is unity. So that (32) simplify to:

$$\ddot{z} + \frac{1}{G}(\partial_{z} G)\ \dot{z}^{2} + \frac{1}{G}(\partial_{z^{*}} G)\dot{z}\dot{z}^{*} = 0\,,$$

$$\ddot{z}^{*} + \frac{1}{G^{*}}(\partial_{z^{*}} G^{*})\ \dot{z^{*}}^{2} + \frac{1}{G^{*}}(\partial_{z} G)\dot{z}\dot{z}^{*} = 0 \tag{33}$$

Also, from the Robertson-Walker like metric (25), the moduli field equations:

$$\ddot{z} + \frac{1}{G}(\partial_{z} G)\ \dot{z}^{2} + \frac{1}{G}(\partial_{z^{*}} G)\dot{z}\dot{z}^{*} = 0\,, \tag{34}$$

$$\ddot{z}^{*} + \frac{1}{G^{*}}(\partial_{z^{*}} G^{*})\ \dot{z^{*}}^{2} + \frac{1}{G^{*}}(\partial_{z} G)\dot{z}\dot{z}^{*} = 0\,. \tag{35}$$

Solving the moduli field equation Equ. (34) with Equ. (27), gives the moduli's variation with time:

$$z(t) \simeq C + \frac{0.001 + 0.25\, t}{(1 + 250\, t)^{0.6}}\,. \tag{36}$$

For $\dot{z}[0] = 0$. C is the integration constant. We take $z[0] = 1$, and $C \sim 1$. Also, we have made a further approximation here by considering the moduli real. In Fig. (2- left) and (2- right) the moduli and the moduli velocity are plotted versus time, respectively. The K$\ddot{a}$hler metric can be directly obtained by substituting the solution of $\dot{z}$ in Equ. (27). In Fig. (3left) one component of the metric $G_{i\bar{j}}$ multiplied by a factor 10^{-2} is plotted versus time. Generally speaking, the K$\ddot{a}$hler potential is given by:

$$\kappa = \ln(1 + z_{i} z^{\bar{j}}) = \ln\left(1 + \delta_{i\bar{j}} z^{i} z^{\bar{j}}\right), \tag{37}$$

in which the Kähler metric Equ. (6) has been driven. We will use Equ. (6) to get κ as a function in time. According to our approximation, it can be written as:

$$G = \partial_{z}\, \partial_{z^{*}}\, \kappa\,. \tag{38}$$

Let all fields depend on time, it becomes:

$$G(t) = \frac{\partial}{\partial t}\left(\frac{1}{\dot{z}}\, \frac{\dot{k}}{\dot{z}^{*}}\right). \tag{39}$$

Solving for the Kähler potential, yields:

$$\kappa\,(t) = -(0.45\, t + 0.008\, t^{2}). \tag{40}$$

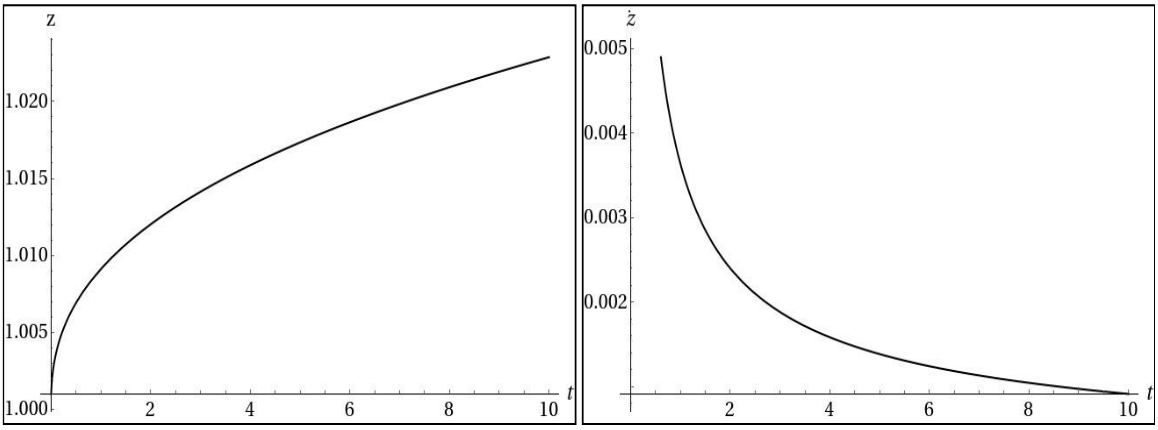

Fig. (2). (Left panel): The moduli is plotted versus time. (Right panel): The moduli velocity is plotted versus time for C=1, and in case of radiation, dust and Λ filled brane with $\Lambda = 1$, and $\widetilde{\Lambda} = 0$.

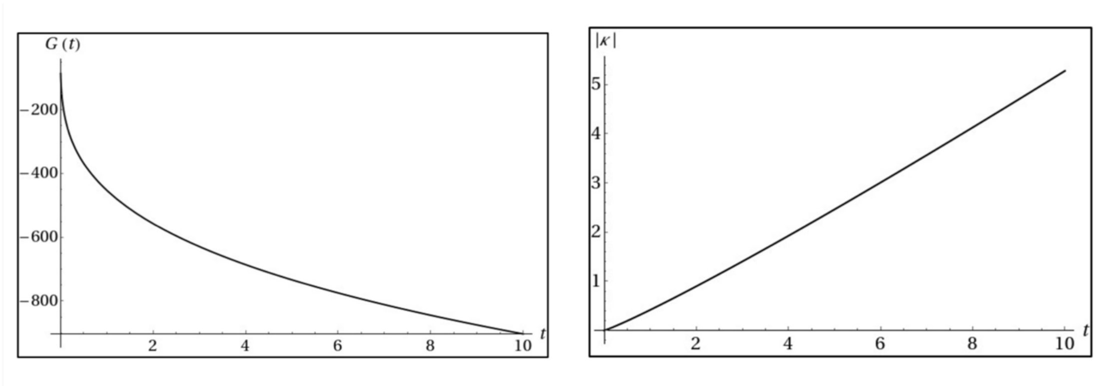

Fig. (3). (Left panel): One component of the Kähler metric multiplied by a factor 10^{-2} is plotted versus time. (Right panel): The modules of the Kähler potential of M_C is plotted versus time.

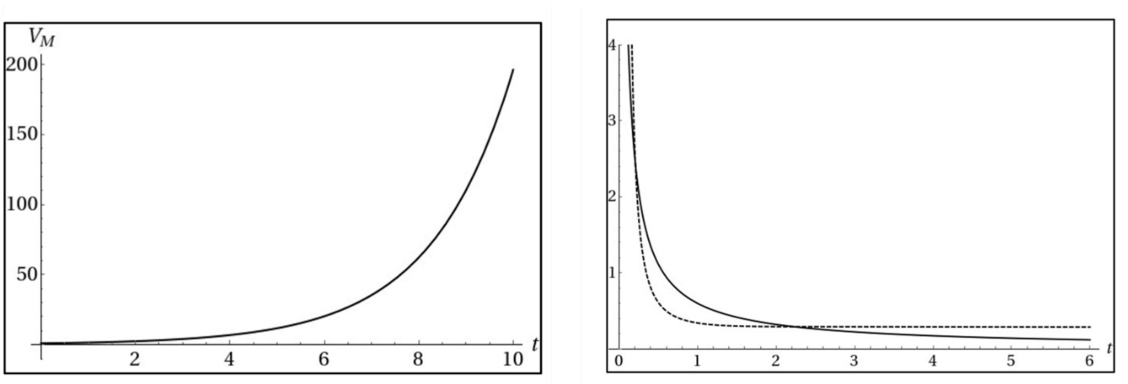

Fig. (4). (Left panel): The volume of the Calabi-Yau manifold plotted versus time. The Hodge number $h_{2,1} = 1$. (Right panel): Numeric and analytic $G_{i\bar{j}}\, z^i z^{\bar{j}}$ versus time in solid and dashed lines, respectively. The Hodge number $h_{2,1} = 1$.

Fig. (3- right) shows the absolute value of the potential plotted versus time. According to Equ. (4) the volume of the CY manifold M can be obtained as long as the Kähler potential is known. Fig. (4- left) shows the volume of the Calabi-Yau manifold plotted versus time. As seen, it increases with time. For the sake of comparison, Fig. (4- right) shows $G_{i\bar{j}}\, z^{i} z^{\bar{j}}$ plotted versus time as it's obtained directly from the numeric solution of the field equations (26) without any approximations a long a side as it's obtained when solving Equ. (27) with Equ. (36).

3. CONCLUSION

Exploring the non-trivial topology of the Calabi-Yau manifold is still a highly demanding quest in theoretical physics. The importance of the Calabi-Yau manifold arises from its vital role in the compactification of many higher dimensional theories. Like when compactifying D = 11 supergravity to N = 2, D = 5 supergravity over CY 3-fold. In this work we have studied a 3-brane embedded in the bulk of D = 5 supergravity, we have solved the Friedmann-like equations in the case of a brane filled with radiation, dust, and energy. We have shown that in this case, the time evolution of the 3-brane coincides with the time evolution of our universe, where the moduli's velocity norm $G_{i\bar{j}}\, z^{i} z^{\bar{j}}$ strongly correlates to the scale factor of the brane universe. That means the cosmology of our universe can be interpreted only by the effects of the bulk of a higher dimensional theory. This explanation needs more analysis and initial conditions to be verified which we keep to a further study. We then used the solutions to explore the complex structure space of the CY manifold. Since we knew the time behavior of $G_{i\bar{j}}\, z^{i} z^{\bar{j}}$, we have solved the moduli field equation to get the time dependence of the complex structure moduli, the Kähler metric, the Kähler potential, and the volume of the Calabi-Yau manifold. That's for a Hodge number $h_{2,1} = 1$, which means the dimensions of M_C are unity and there is a single moduli, since z^i are considered the coordinates that define M_C. The time dependence of the Kähler potential is negative like the Kähler metric. The absolute value of the potential increases with time. Also, the volume of the CY manifold infinitely increases with time which is a deduction that the brane- world and the bulk are keeping expanding with time.

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تطبيقات الجاذبية الفائقة N=2 , D=5 على طوبولوجيا نسيج Calabi-Yau

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الملخص

عند تقليل أبعاد الجاذبية الفائقة N=1 , D=11 إلى الجاذبية الفائقة N=2 , D=5 باستخدام نسيج Calabi-Yau فإن لاجرانج النظرية الجديدة يعتمد تماما على طوبولوجيا نسيج Calabi-Yau ، حرفيا يعتمد على حقول كمية تسمى Moduli تعرف بأنها احداثيات نسيج Calabi-Yau المعقد . عند حل معادلات المجال للنظرية فإننا نستطيع إيجاد سلوك ال Moduli مع الزمن وبالتالي إيجاد جميع خصائص نسيج Calabi-Yau ، أي إيجاد السلوك الزمني لل metric الخاصة به والسلوك الزمني لل potentiel الخاص به وكذلك السلوك الزمني لحجم النسيج. في هذا السياق ندرس كوننا المعروف كسطح ثلاثي الأبعاد (مكانية) موجود في زمكان خماسي الأبعاد. لحل معادلات المجال فإننا اعتبرنا هذا السطح الثلاثي ملئ بإشعاع ومادة عادية وطاقة تسبب في تمدد متسارع لهذا السطح تماما مثل التطور الكوني لعالمنا.