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Some Properties of Weighted Norm Inequalities for the Hardy-Littlewood Maxima Function

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Research Article

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Abstract

Many researchers have expended their efforts on Hardy-Littlewood Maxima operator but little or no work has been done if the operator is acting on a power function.
In this article, new characterization of Hardy-Littlewood Maxima operator bounded from $L^p(\mathbb{R}^n, wdx)$ to $L^q(\mathbb{R}^n, vdx)$ for weight functions $v(x)$ and some non-trivial $w(x)$ are proved. Our novel methods generalized and sharpened Wo-sang Young results to perfection.

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1 Introduction

Maximal functions are usually expressed in term of the suprimum, with respect to parameters, classes of operators, mappings, or transformations. For any given Lebesgue measurable function $f(x)$ on $\mathbb{R}$, the maximal functions are:

$M^+f(x) = \sup\{(u-x)^{-1} \left| \int_x^u f(t)dt \right| : u > x\}$ and
$M^-f(x) = \sup\{(x-u)^{-1} \left| \int_u^x f(t)dt \right| : x > u\}$.

Also, we let $Mf(x) = \max\left(T^+f(x), T^-f(x)\right) = \sup\{(u-x)^{-1} \left| \int_x^u f(t)dt \right| : u \neq x\}$ and T be the Hardy-Littlewood maximal operator defined by

$$Tf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f|\, dy \qquad (1.1)$$

where $B(x,r)$ is the ball of radius r centered at x and $|B(x,r)|$ is its Lebesgue measure.

Rubio de Francia (1981) observed the following:

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THEOREM 1.1

Given $v \geq 0$ and $1 < p < \infty$, the following conditions (a) and (b) are equivalent:

(a) There is $w < \infty$ almost everywhere such that

$$\int_{\mathbb{R}^n} |Tf|^p v dx \leq C \int_{\mathbb{R}^n} |f|^p w dx \qquad (1.2)$$

for all $f \in L^p(\mathbb{R}^n, wdx)$.

(b) $u \in L^1_{loc}(\mathbb{R}^n)$ and $(\int_{x \leq R} u)^{\frac{1}{p}} = O(\mathbb{R}^n) \qquad (\mathbb{R} \to \infty)$.

He conjectured that (b) is sufficient for (a). Similar result was obtained by Carleson and Jone (1981) and Fefferman and Stein (1971). Also, Wo-sang Young (1982) obtained the necessary and sufficient condition for (a) and proved the following result:

THEOREM 1.2

Given $v \geq 0$ and $1 < p < \infty$, there is $w < \infty$ almost everywhere such that

$$\int_{\mathbb{R}^n} |Tf|^p v dx \leq C \int_{\mathbb{R}^n} |f|^p w dx \qquad (1.3)$$

is equivalent to

$$\int_{\mathbb{R}^n} \frac{v(x)}{\left(1 + |x|^n\right)^p} dx < \infty$$

for all $f \in L^p(\mathbb{R}^n, wdx)$, where C denotes a constant depending only on n and p.

Kerman and Sawyer (1989), Oguntuase and Adegoke (1999), Rauf and Imoru (2002) and Rauf and Omolehin (2006) obtained some generalization of weighted norm inequalities for integral operators.

In this paper, we obtain necessary and sufficient condition on weight function $v \geq 0$ such that the Hardy-Littlewood maximal operator is bounded from $L^p(\mathbb{R}^n, wdx)$ to $L^q(\mathbb{R}^n, vdx)$ and $1 < p \leq q < \infty$ for some $w < \infty$ almost everywhere. Occasionally, we shall use $\boldsymbol{x}$ to replace $x_1 \cdots x_n$, $\boldsymbol{dx}$ to replace $dx_1 \cdots dx_n$ and similarly for $\boldsymbol{y}$ and $\boldsymbol{dy}$.

2 Main Results

The statement of the results is as follows:

THEOREM 2.1

Let $v \geq 0$ and suppose T is Hardy-Littlewood maximal operator defined by (1.1). For $p(1, \infty)$ and $p^{-1} + q^{-1} = 1$, there exists $w(x) < \infty$ almost everywhere and $C(n, p, q) > 0$ such that:

$$\int_{\mathbb{R}^n} |Tf(x_1 \cdots x_n)|^q \sum_{i=1}^{n} v_i dx_1 \cdots dx_n \leq C \int_{\mathbb{R}^n} |f(x_1 \cdots x_n)|^p w_i dx_1 \cdots dx_n \tag{2.1}$$

for any given convex function $f \in L^p(\mathbb{R}^n, wdx)$, if and only if

$$\int_{\mathbb{R}^n} \frac{v_i(x_1 \cdots x_n)}{\left(1 + |x_1 \cdots x_n|^n\right)^{p-1}} dx_1 \cdots dx_n < \infty \tag{2.2}$$

The inequality (2.1) holds with v_i equal to (2.2).

Proof:

Let A denotes a set with positive measure for which w is bounded almost everywhere. Suppose $A \subset \{|\boldsymbol{x}| \leq M\}$ for all $1 < M < \infty$ and let $f = \chi_{A(x_1 \cdots x_n,\ y_1 \cdots y_n)}$.

Then,

$$\int_{\mathbb{R}^n} |Tf(x_1 \cdots x_n)|^q \sum_{i=1}^{n} v_i dx_1 \cdots dx_n = \int_{\mathbb{R}^n} (\chi_{A(x_1 \cdots x_n,\ y_1 \cdots y_n)})^q w_i dx_1 \cdots dx_n$$

$$\leq C \int_{\mathbb{R}^n} |f(x_1 \cdots x_n)|^p w_i dx_1 \cdots dx_n$$

by (1.2)

$$= \int_A w_i < \infty$$

for $|x_1 \cdots x_n| > M$, we let $I = (Tf)(x_1 \cdots x_n)$.

Then,

$$I = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{\mathrm{B(x,r)}} |f(y_1 \cdots y_n)|\, dy_1 \cdots dy_n \geq C \frac{|A|}{|x_1 \cdots x_n|^n}$$

Hence,

$$\int_{\mathbb{R}^n} |I|^q \sum_{i=1}^{n} v_i dx_1 \cdots dx_n = \int_{\mathbb{R}^n} |Tf(x_1 \cdots x_n)|^q \sum_{i=1}^{n} v_i dx_1 \cdots dx_n$$

$$= \int_{\mathbb{R}^\mathrm{n}} \left(\sup_{r>0} \frac{1}{|B(x,r)|} \int_{\mathrm{B(x,r)}} |f(y_1 \cdots y_n)|\, dy_1 \cdots dy_n \right)^{\mathrm{q}} \sum_{i=1}^{n} v_i dx_1 \cdots dx_n$$

$$> \left(C \frac{|A|}{M^n} \right)^q \int_{\mathbb{R}^n} \frac{\sum_{i=1}^{n} v_i(x_1 \cdots x_n)}{\left(1 + |x_1 \cdots x_n|^n\right)^{p-1}} dx_1 \cdots dx_n$$

$$> \left(C \frac{|A|}{M^n} \right)^p \int_{\mathbb{R}^n} \frac{v_i(x_1 \cdots x_n)}{\left(1 + |x_1 \cdots x_n|^n\right)^{p-1}} dx_1 \cdots dx_n$$

Since the left hand side of this inequality is finite, so also the right hand side.

Hence,

$$\int_{\mathbb{R}^n} \frac{v_i(x_1 \cdots x_n)}{\left(1+|x_1 \cdots x_n|^n\right)^{p-1}} dx_1 \cdots dx_n < \infty$$

which shows that (2.1) implies (2.2).

Conversely, let

$$v_i = \begin{cases} v(x_i) & \text{if } v(x_i) \geq 1 \\ \\ i & \text{if } v(x_i) < 1 \end{cases}$$

for all $i \in \mathbb{N}$

Then,

$$\int_{\mathbb{R}^n} \frac{\sum_{i=1}^n v_i(x_1 \cdots x_n)}{\left(1+|x_1 \cdots x_n|^n\right)^{p-1}} dx_1 \cdots dx_n \leq \int_{\mathbb{R}^n} \frac{(v_1(x_1)+1)+\cdots+(v_n(x_n)+n)}{\left(1+|x_1 \cdots x_n|^n\right)^{p-1}} dx_1 \cdots dx_n$$

$$= \int_{\mathbb{R}^n} \frac{\left(\sum_{i=1}^n (v_i(x_i)+i)\right)}{\left(1+|x_1 \cdots x_n|^n\right)^{p-1}} dx_1 \cdots dx_n$$

$$\leq \int_{\mathbb{R}^n} \frac{\sum_{i=1}^n v_i(x_i)}{\left(1+|x_1 \cdots x_n|^n\right)^{p-1}} dx_1 \cdots dx_n + \int_{\mathbb{R}^n} \frac{\sum_{i=1}^n i}{\left(1+|x_1 \cdots x_n|^n\right)^{p-1}} dx_1 \cdots dx_n \tag{2.3}$$

Each term of (2.3) is finite, since $v \leq \sum_{i=1}^n v_i$ and therefore (2.3) is finite.

On the other hand, assume $|\boldsymbol{x}| \leq \mathbb{R}$ we have, $I \geq C \frac{|A|}{\mathbb{R}^n}$. We need to establish that there is a weight function $w(x)$ finite almost everywhere such that:

$$\int_{\mathbb{R}^n} |Tf(x_1 \cdots x_n)|^q \sum_{i=1}^n v_i dx_1 \cdots dx_n \leq C \int_{\mathbb{R}^n} |f(x_1 \cdots x_n)|^p w_i dx_1 \cdots dx_n \tag{2.4}$$

for all convex function $f \in L^p(\mathbb{R}^n, wdx)$.

Let $u(\boldsymbol{x}) = \left(1+|x_1 \cdots x_n|^n\right)^{2-p}$ and take $T(u \sum_{i=1}^n v_i) < \infty$ almost everywhere, then, to see this, we shall first consider $\{|\boldsymbol{x}| \leq \mathbb{R}\}$. For any such $\boldsymbol{x}$, we have

$$\sup_{r_i \leq \mathbb{R}} \frac{1}{r_i^{\mathrm{n}}} \int_{|y_i - x_i| \leq r_i} u \sum_{i=1}^n v_i \leq \mathrm{C}T\left(\sum_{i=1}^n v_i \chi_{\{|y_i \cdots x_i)| \leq \mathbb{R}\}}\right)(x)$$

$$= \mathrm{C}T\left(\sum_{i=1}^n v_i \chi_{\{|y_i| \leq 2\mathbb{R}\}}\right)(x).$$

We claim that $T\left(\sum_{i=1}^n v_i \chi_{\{|y_i| \leq 2\mathbb{R}\}}\right)(x)$ is finite almost everywhere, since $\sum_{i=1}^n v_i \in L_{loc}^1(\mathbb{R}^n)$

Next assume $r > M \in \mathbb{R}$, we have

$$\sup_{r_i \leq \mathbb{R}} \frac{1}{r_i^{\mathrm{n}}} \int_{|y_i - x_i| \leq r_i} u \sum_{i=1}^n v_i \leq \sup_{r_i \leq \mathbb{R}} \frac{1}{r_i^{\mathrm{n}}} \int_{|y_i| \leq 2r_i} \frac{\sum_{i=1}^n v_i(y_i)}{\sqrt[2-p]{u(y)^{p-2}}} dy_1 \cdots dy_n$$

$$\leq C\int_{\mathbb{R}^n}\frac{\sum_{i=1}^{n}v_i(y_i)}{\sqrt[2-p]{u(y)^{p-1}}}dy_1\cdots dy_n$$
$$\leq C'\int_{\mathbb{R}^n}\frac{\sum_{i=1}^{n}v_i(y_i)}{\sqrt[2-p]{u(y)^{p}}}dy_1\cdots dy_n$$

where $C' \geq C\left(1+|\boldsymbol{y}|^n\right)$.

Hence, $T(u_i\sum_{i=1}^{n}v_i) < \infty$ almost everywhere. Now, let $w_i = u_i^{-3}T(u_i\sum_{i=1}^{n}v_i)$. Then, $w_i < \infty$ almost everywhere. It is sufficient to show that (2.4) holds for this $w_i(x)$.

Let $f \in L^p(\mathbb{R}^n, wdx)$, and for $k \in \{\mathbb{N}\cup 0\}$ and $f_k = f_i\chi_{\{2^k\leq|x|<2^{k+1}\}}$, the quantity $(s-x)^{-1}\int_x^s f(\boldsymbol{y})\boldsymbol{dy}$ is continuous on s on the real interval (x,∞), it follows that $\lambda_{T^*f}(x)$ is an open set for any particular positive real number x. Also, let $\{(a_k(x)),(b_k(x))\}$ be the class of disjoint open interval such that $\lambda_{T^+f}(x) = Y_{k=1}^{\infty}(a_k,b_k)$ and $A \subset \{|x| \leq \mathbb{R}\}$ then we have $\lambda_{Tf}(x) \subseteq \lambda_{T^+f}(x)\lambda_{T^-f}(x)$.

Hence,

$$m\lambda_{Tf}(x) \leq m\lambda_{T^+f}(x) + m\lambda_{T^-f}(x)$$
$$= x^{-1}\left(\int_{\lambda_{T^+f}(x)}f(y)dy + \int_{\lambda_{T^-f}(x)}f(y)dy\right)$$
$$\leq 2x^{-1}\int_{\lambda_{Tf}(x)}f(y)dy$$

In addition, we have

$$\int_{|x|\leq 2^{k+2}}|Tf_k|^q\sum_{i=1}^{n}v_i\,dy \leq C2^{kn(p-1)}\int_{\mathbb{R}^n}|Tf_k|^p u_i\sum_{i=1}^{n}v_i\,dy$$
$$\leq C2^{kn(p-1)}\int_{\mathbb{R}^n}|f_k|^p\,T\left(u_i\sum_{i=1}^{n}v_i\right)dy$$
$$\leq C2^{-2kn(p-1)}\int_{\mathbb{R}^n}|f_k|^p\,w_i dy$$

Thus, for $|\boldsymbol{x}| > 2^{k+1}$, we have,

$$Tf_k(\boldsymbol{x}) \leq \frac{C}{|\boldsymbol{x}|^n}\int_{2^k\leq|x|<2^{k+1}}|f|$$
$$= \frac{C}{|\boldsymbol{x}|^n}\int_{2^k\leq|x|<2^{k+1}}|f|w_i^{\frac{1}{q}}w_i^{-\frac{1}{q}}$$
$$\leq \frac{C}{|\boldsymbol{x}|^n}\left(\int_{\mathbb{R}^n}|f_k|^p\,w_i^{p-1}\right)^{\frac{1}{p}}\left(\int_{2^k\leq|x|<2^{k+1}}w_i^{-1}\right)^{\frac{1}{q}}$$

By Hȯlder's inequality, we have,

$$\int_{|x|>2^{k+2}}|Tf_k|^q\sum_{i=1}^{n}v_i \leq C\left(\int_{\mathbb{R}^n}\frac{\sum_{i=1}^{n}v_i(\boldsymbol{x})}{\sqrt[2-p]{u(\boldsymbol{x})^p}}d\boldsymbol{x}\right)\left(\int_{\mathbb{R}^n}|f|^p\,w_i^{p-1}\right)\left(\int_{2^k\leq|x|<2^{k+1}}w_i^{-1}\right)^{p-1}$$

To estimate the last factor, we observe that for $2^k \le |x| < 2^{k+1}$, which means that

$$T\left(u_i \sum_{i=1}^{n} v_i\right)(\boldsymbol{x}) \ge \frac{C}{2^{n(k+1)}} \int_{|y|\le 2^k} \frac{\sum_{i=1}^{n} v_i(\boldsymbol{y})}{\sqrt[2-p]{u(\boldsymbol{y})^{p-1}}} \boldsymbol{dy}$$
$$\ge \frac{C}{2^{n(k+1)}} \int_{|y|\le 2^k} \frac{\sum_{i=1}^{n} v_i(\boldsymbol{y})^{2-p}\sqrt{u(\boldsymbol{y})}}{\sqrt[2-p]{u(\boldsymbol{y})^{p-1}}} \boldsymbol{dy}$$
$$\ge 2^{knp(1-p)} * 2^{-kn}$$

which implies that $w(\boldsymbol{x}) \ge C 2^{knp(1-p)} * 2^{-kn}$ and $\left(\int_{2^k \le |x| < 2^{k+1}} w^{1-p}\right)^{p-1} \le 2^{knp(p-1)}$

Hence,

$$\int_{\mathbb{R}^n} |Tf_k|^q \sum_{i=1}^{n} v_i \le C\left[\int_{\mathbb{R}^n} \frac{v_i(\boldsymbol{x})}{\sqrt[2-p]{u(\boldsymbol{x})^{p-1}}} d\boldsymbol{x} + \mathbf{1}\right]\left(\int_{\mathbb{R}^n} |f|^p\, w_i^{1-p}\right)\left(2^{knp(1-p)} * 2^{-kn}\right)$$

For $|x| > 2^{k+2}$, we have

$$\int_{\mathbb{R}^n} |Tf_k|^q \sum_{i=1}^{n} v_i \le \left[\sum_{k=1}^{n}\left(\int_{\mathbb{R}^n} |Tf_k|^p \sum_{i=1}^{n} v_i\right)^{\frac{1}{p}}\right]^p$$

By Minkowski's inequality

$$\le C\left[\int_{\mathbb{R}^n} \frac{v_i(\boldsymbol{x})}{\sqrt[2-p]{u(\boldsymbol{x})^{p-1}}} d\boldsymbol{x} + \mathbf{1}\right]\left(\int_{\mathbb{R}^n} |f|^p\, w_i^{1-p}\right)\left[\sum_{k=0}^{\infty} 2^{-kn}\right]^p$$
$$\le C\left[\int_{\mathbb{R}^n} \frac{v_i(\boldsymbol{x})}{\sqrt[2-p]{u(\boldsymbol{x})^{p-1}}} d\boldsymbol{x} + \mathbf{1}\right]\left(\int_{\mathbb{R}^n} |f|^p\, w_i\right)$$

which prove the sufficient aspect of the theorem and hence the results are valid for all $n \in \mathbb{N}$ and the proof is complete.

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Competing interests

Authors have declared that no competing interests exist.

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