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Applications to the Quadratic Forms and Product-to-sum Formulae

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Abstract

In this paper, we attempt to express two values

$$G(q) := \sum_{n=1}^{\infty} g(n)q^n = 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32}(1-q^n)^{16}}{(1+q^{2n})^{12}},$$
$$H(q) := \sum_{n=1}^{\infty} h(n)q^n = 2^{12} q^3 \prod_{n=1}^{\infty} \left(1+q^{2n}\right)^4 \left(1-q^{4n}\right)^{16}$$

into sums of squares and obtain

$$g(n) = 8 \sum_{\substack{(x_1,x_2,\cdots,x_{11},x_{12})\in\mathbb{Z}^{12} \\ x_1^2+x_2^2+\cdots+x_9^2+x_{10}^2+2(x_{11}^2+x_{12}^2)=n}} \left(x_{10}^2 - 2x_{12}^2\right),$$
$$h(n) = 1024 \sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4 \\ 2(x_1^2+x_2^2+2x_4^2)+x_3^2=n}} (-1)^{x_1^2+x_2^2+1} \left(x_1^4 - 3x_1^2x_2^2\right)\left(x_3^2 - 4x_4^2\right)$$

(see Theorem 1.1 and Theorem 1.2, respectively). Finally, we obtain some relations $\varphi_1(q)$ according to $\xi_1(q)$, $\varphi(q)$, and $\psi(q)$.

Keywords: Quadratic forms; Theta functions; Eisenstein series; Infinite products; Product-to-sum formulae

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1 Introduction

The set of positive integers is denoted by $\mathbb{N}$ and the set of nonnegative integers by $\mathbb{N}_0$. Throughout this paper $q \in \mathbb{C}$ is taken to satisfy $|q| < 1$, for such q we define

$$E_k = E_k(q) := \prod_{n\in\mathbb{N}} \left(1 - q^{kn}\right), \qquad k \in \mathbb{N}.$$

As following the K. S. Williams' paper [1], we define

$$\widetilde{\sigma}_k(n) := \begin{cases} \sum_{\substack{d\in\mathbb{N} \\ d|n \\ n/d \text{ odd}}} d^k, & \text{if } n \in \mathbb{N}, \\ 0, & \text{if } n \in \mathbb{Q},\, n \notin \mathbb{N}, \end{cases}$$

for $k \in \mathbb{N}_0$ and $n \in \mathbb{Q}$. The Eisenstein series $\xi_k(q)$ is defined for $k \in \mathbb{N}$ with $k \equiv 1 \pmod 2$ by

$$\xi_k(q) := \sum_{n=1}^{\infty} \widetilde{\sigma}_k(n) q^n = \sum_{n=1}^{\infty} \frac{n^k q^n}{1 - q^{2n}}. \tag{1.1}$$

The one-dimensional theta function $\varphi_k(q)$ is defined for $k \in \mathbb{N}_0$ by

$$\varphi_k(q) := \sum_{n=-\infty}^{\infty} n^{2k} q^{n^2} \qquad \text{and} \qquad \varphi(q) := \varphi_0(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

We also require the theta function

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

For example Cooper introduces the famous identity

$$q\psi^8(q) = \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^{2n}}$$

in ([2], Eq. (3.71), p. 136), therefore, by (1.1), we have

$$\xi_3(q) = q\psi^8(q). \tag{1.2}$$

Next we define the two-dimensional theta function $\Phi_{k,l,m}(q)$ by

$$\Phi_{k,l,m}(q) := \sum_{r,s=-\infty}^{\infty} \left(r\sqrt{l} + s\sqrt{-m}\right)^{2k} q^{lr^2+ms^2}, \qquad k, l, m \in \mathbb{N}, \tag{1.3}$$

and the quantities

$$\begin{aligned} A_{l,m}(q) &:= l\xi_1(-q^l) - m\xi_1(-q^m), \\ B_{l,m}(q) &:= l^2\xi_3(-q^l) + m^2\xi_3(-q^m). \end{aligned} \tag{1.4}$$

In this paper, we attempt to express two values

$$G(q) := \sum_{n=1}^{\infty} g(n)q^n = 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32}(1-q^n)^{16}}{(1+q^{2n})^{12}} \tag{1.5}$$

and

$$H(q) := \sum_{n=1}^{\infty} h(n)q^n = 2^{12} q^3 \prod_{n=1}^{\infty} \left(1+q^{2n}\right)^4 \left(1-q^{4n}\right)^{16} \tag{1.6}$$

into sums of squares based on K. S. Williams' paper [1]. Here $G(q)$ and $H(q)$ are first defined in the paper [3] to obtain some convolution sum formulae related to $\sum_{m<n/8} \sigma_3(m)\sigma_3(n-8m)$.

In Sections 2 and 3 we obtain as following results :

Theorem 1.1. *Let* $n \in \mathbb{N}$. *Then we have*

$$g(n) = 8 \sum_{\substack{(x_1,x_2,\cdots,x_{11},x_{12})\in\mathbb{Z}^{12} \\ x_1^2+x_2^2+\cdots+x_9^2+x_{10}^2+2(x_{11}^2+x_{12}^2)=n}} \left(x_{10}^2 - 2x_{12}^2\right).$$

Theorem 1.2. *Let* $n \in \mathbb{N}$. *Then we have*

$$h(n) = 1024 \sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4 \\ 2(x_1^2+x_2^2+2x_4^2)+x_3^2=n}} (-1)^{x_1^2+x_2^2+1} \left(x_1^4 - 3x_1^2x_2^2\right)\left(x_3^2 - 4x_4^2\right).$$

Finally, in Section 4 we obtain some properties of $\varphi_1(q)$ according to $\xi_1(q)$, $\varphi(q)$, and $\psi(q)$:

Lemma 1.3. *For* $q \in \mathbb{C}$ *with* $|q| < 1$ *we obtain*

(a)

$$\varphi_1(q)\xi_1(q) + \varphi_1(-q)\xi_1(-q) = -4\xi_1(q)\xi_1(-q)\varphi(q^4),$$

(b)

$$\varphi_1(q)\xi_1(q) - \varphi_1(-q)\xi_1(-q) = -8q\xi_1(q)\xi_1(-q)\psi(q^8),$$

(c)

$$\varphi_1^2(q)\xi_1^2(q) - \varphi_1^2(-q)\xi_1^2(-q) = 32q\xi_1^2(q)\xi_1^2(-q)\psi^2(q^4),$$

(d)

$$\varphi_1(q)\varphi(-q) + \varphi_1(-q)\varphi(q) = 4\varphi(-q^2)\varphi_1(-q^2),$$

(e)

$$\varphi_1(q)\varphi(-q) - \varphi_1(-q)\varphi(q) = 4q\varphi^4(-q^4)\psi^2(q^4),$$

(f)

$$\varphi_1^2(q)\varphi^2(-q) - \varphi_1^2(-q)\varphi^2(q) = 16q\varphi(-q^2)\varphi^4(-q^4)\psi^2(q^4)\varphi_1(-q^2).$$

2 Proof of Theorem 1.1

We can find Proposition 2.1 and its proof as the main theorem in [1] :

Proposition 2.1. *(See [1] Theorem 1.1) Let $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$. Let $r, s, t, u \in \mathbb{N}_0$ be such that*

$$r + s + t + u = k. \tag{2.1}$$

Let $v, w, x, y \in \mathbb{N}_0$ be such that

$$v + w + x + y = l. \tag{2.2}$$

Set

$$m = k + 2l \tag{2.3}$$

so that $m \in \mathbb{N}$ and $m \geq 2$. Let

$$\begin{aligned} P(x_1, \cdots, x_m) = \frac{1}{2^l} \prod_{g=r+1}^{r+v} \left(x_g^2 - 2x_{g+s+l+y}^2\right) \prod_{g=r+v+1}^{r+v+w} \left(x_g^2 - 3x_{g+s+t+l+y}^2\right) \\ \times \prod_{g=r+v+w+1}^{r+v+w+x} \left(x_g^2 - 4x_{g+s+t+l+y+u}^2\right) \prod_{g=r+v+w+x+1}^{r+l} \left(x_g^4 - 3x_g^2 x_{g+y}^2\right) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} Q(x_1, \cdots, x_m) = x_1^2 + \cdots + x_{r+l+y}^2 + 2x_{r+l+y+1}^2 \cdots + 2x_{r+s+l+v+y}^2 \\ + 3x_{r+s+l+v+y+1}^2 + \cdots + 3x_{r+s+t+l+v+w+y}^2 \\ + 4x_{r+s+t+l+v+w+y+1}^2 + \cdots + 4x_m^2. \end{aligned} \tag{2.5}$$

Let

$$\begin{aligned} a_1 &= -2r + 2v + 4y, & a_6 &= 5t + 3w, \\ a_2 &= 5r - 2s + v + 3w + 2y, & a_8 &= -2s + 5u + 2v, \\ a_3 &= -2t, & a_{12} &= -2t, \\ a_4 &= -2r + 5s - 2u + v + 6x + 4y, & a_{16} &= -2u. \end{aligned} \tag{2.6}$$

Then, for $n \in \mathbb{N}$ with $n \geq l$, we have

$$\left[q^l E_1^{a_1} E_2^{a_2} E_3^{a_3} E_4^{a_4} E_6^{a_6} E_8^{a_8} E_{12}^{a_{12}} E_{16}^{a_{16}}\right]_n = \sum_{\substack{(x_1, \cdots, x_m) \in \mathbb{Z}^m \\ Q(x_1, \cdots, x_m) = n}} P(x_1, \cdots, x_m) \tag{2.7}$$

and

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 6a_6 + 8a_8 + 12a_{12} + 16a_{16} = 24l,$$

where P is a polynomial in $x_1, \cdots, x_m$ with rational coefficients and Q is a positive-definite, diagonal, quadratic form in $x_1, \cdots, x_m$ with integral coefficients. And K. S. Williams writes

$$[f(q)]_n = f_n, \qquad n \in \mathbb{N}_0,$$

if $f(q) = \sum_{n=0}^{\infty} f_n q^n$.

Using Proposition 2.1 we can obtain Theorem 1.1 easily, but we need more skill for Theorem 1.2 so we calculate it in Section 3 independently.

Proof of Theorem 1.1. Let us change $G(q)$ in (1.5) as

$$\begin{aligned} G(q) &= 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32}(1-q^n)^{16}}{(1+q^{2n})^{12}} \\ &= 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32}(1-q^n)^{16}}{(1+q^{2n})^{12}} \cdot \frac{\left(1-q^{2n}\right)^{12}}{(1-q^{2n})^{12}} \cdot \frac{(1-q^n)^{16}}{(1-q^n)^{16}} \\ &= 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{16}\left(1-q^{2n}\right)^{16}\left(1-q^{2n}\right)^{12}(1-q^n)^{16}}{(1-q^{4n})^{12}(1-q^n)^{16}} \\ &= 2^4 q \prod_{n=1}^{\infty} \frac{\left(1-q^{2n}\right)^{44}}{(1-q^{4n})^{12}(1-q^n)^{16}} \\ &= 2^4 q E_2^{44} E_4^{-12} E_1^{-16}. \end{aligned} \tag{2.8}$$

Then we apply (2.8) to Proposition 2.1 and so, by (2.2), (2.6) and (2.7), we obtain

$$\begin{aligned} l &= v+w+x+y = 1, \\ a_1 &= -2r+2v+4y = -16, \\ a_2 &= 5r-2s+v+3w+2y = 44, \\ a_3 &= -2t = 0, \\ a_4 &= -2r+5s-2u+v+6x+4y = -12, \\ a_6 &= 5t+3w = 0, \\ a_8 &= -2s+5u+2v = 0, \\ a_{12} &= -2t = 0, \\ a_{16} &= -2u = 0. \end{aligned} \tag{2.9}$$

So from $a_3, a_6,$ and a_{16} we can deduce that

$$t=0, \quad w=0, \quad \text{and} \quad u=0. \tag{2.10}$$

Since $u=0$ we have

$$s=v$$

from a_8 and similarly, because $w=0$ we obtain

$$v+x+y=1$$

from l. Also a_1 leads

$$-r+v+2y=-8 \tag{2.11}$$

and since $s=v$ and $w=0$, the condition a_2 follows

$$5r-2s+v+3w+2y = 5r-v+2y = 44. \tag{2.12}$$

Again by using the facts $u = 0, s = v$, and $v + x + y = 1$ the condition a_4 shows that

$$\begin{aligned} &-2r + 5s - 2u + v + 6x + 4y \\ &\quad = -2r + 5v + v + 6x + 4y \\ &\quad = -2r + 6(v + x + y) - 2y \\ &\quad = -2r + 6 - 2y \\ &\quad = -12 \end{aligned}$$

so

$$r + y = 9. \tag{2.13}$$

Then, by (2.13), we can write (2.11) as

$$-r + v + 2y = -r + v + 2(9 - r) = -3r + v + 18 = -8$$

and so

$$v = 3r - 26 \tag{2.14}$$

which shows that

$$r \geq 9 \tag{2.15}$$

since Proposition 2.1 restricts $v, r \in \mathbb{N}_0$. Therefore, by combining (2.15) and $y \in \mathbb{N}_0$, (2.13) concludes that

$$r = 9 \qquad \text{and} \qquad y = 0. \tag{2.16}$$

Sequentially, by (2.16), Eq. (2.14) constructs

$$v = 1 \tag{2.17}$$

and it is obvious

$$s = 1 \tag{2.18}$$

by $s = v$. Lastly, by (2.16) and (2.17), the condition $v + x + y = 1$ leads

$$x = 0. \tag{2.19}$$

Now all values also confirm Eq. (2.12). Let us find k and m :

$$\begin{aligned} k &= r + s + t + u = 10, \\ m &= k + 2l = 12, \end{aligned}$$

where we used (2.1), (2.3), (2.9), (2.10), (2.16), and (2.18). So (2.4) becomes

$$P(x_1, \cdots, x_{12}) = \frac{1}{2}\left(x_{10}^2 - 2x_{12}^2\right)$$

and (2.5) constructs

$$Q(x_1, \cdots, x_{12}) = x_1^2 + \cdots + x_{10}^2 + 2x_{11}^2 + 2x_{12}^2,$$

thus these polynomials deduce (2.7) as

$$\left[qE_1^{-16}E_2^{44}E_4^{-12}\right]_n = \sum_{\substack{(x_1,\cdots,x_{12})\in\mathbb{Z}^{12}\\ x_1^2+\cdots+x_{10}^2+2x_{11}^2+2x_{12}^2=n}} \frac{1}{2}\left(x_{10}^2-2x_{12}^2\right).$$

Therefore we can write (2.8) as

$$\begin{aligned} G(q) = \sum_{n=1}^{\infty} g(n)q^n &= 2^4 \sum_{n=1}^{\infty} \left[qE_1^{-16}E_2^{44}E_4^{-12}\right]_n q^n \\ &= 2^3 \sum_{\substack{(x_1,x_2,\cdots,x_{11},x_{12})\in\mathbb{Z}^{12}\\ x_1^2+x_2^2+\cdots+x_9^2+x_{10}^2+2(x_{11}^2+x_{12}^2)=n}} \left(x_{10}^2-2x_{12}^2\right)q^n. \end{aligned}$$

□

3 Proof of Theorem 1.2

As mentioned in Section 2, we can not use Proposition 2.1 to express $H(q)$ as sums of squares :
Remark 3.1. From (1.6) we have

$$\begin{aligned} H(q) &= 2^{12}q^3\prod_{n=1}^{\infty}\left(1+q^{2n}\right)^4\left(1-q^{4n}\right)^{16} \\ &= 2^{12}q^3\prod_{n=1}^{\infty}\left(1+q^{2n}\right)^4\cdot\frac{\left(1-q^{2n}\right)^4}{\left(1-q^{2n}\right)^4}\cdot\left(1-q^{4n}\right)^{16} \\ &= 2^{12}q^3\prod_{n=1}^{\infty}\left(1-q^{4n}\right)^{20}\left(1-q^{2n}\right)^{-4} \\ &= 2^{12}q^3E_4^{20}E_2^{-4}. \end{aligned} \tag{3.1}$$

Then, applying (3.1) to (2.7) in Proposition 2.1, we obtain (2.2) and (2.6) as

$$\begin{aligned} &l = v+w+x+y = 3, \\ &a_1 = -2r+2v+4y = 0, \\ &a_2 = 5r-2s+v+3w+2y = -4, \\ &a_3 = -2t = 0, \\ &a_4 = -2r+5s-2u+v+6x+4y = 20, \\ &a_6 = 5t+3w = 0, \\ &a_8 = -2s+5u+2v = 0, \\ &a_{12} = -2t = 0, \\ &a_{16} = -2u = 0. \end{aligned} \tag{3.2}$$

Thus from a_3, a_6, a_8, and a_{16} in (3.2) we can observe that

$$t = 0, \qquad w = 0, \qquad u = 0, \qquad \text{and} \qquad s = v.$$

So since $w = 0$, l in (3.2) shows that

$$v + w + x + y = v + x + y = 3. \tag{3.3}$$

Also because $u = 0$ and $s = v$, a_4 leads that

$$\begin{aligned} &-2r + 5s - 2u + v + 6x + 4y \\ &\quad = -2r + 5v + v + 6x + 4y \\ &\quad = -2r + 6(v + x + y) - 2y \\ &\quad = 20 \end{aligned}$$

so

$$-r + 3(v + x + y) - y = 10. \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$r + y = -1,$$

which contradicts that $r, y \in \mathbb{N}_0$ as Proposition 2.1 suggests.

By the reason of Remark 3.1, we approach $H(q)$ in another way. Now, the infinite product representations of $\varphi(\pm q)$ and $\psi(\pm q)$ due to Jacobi are given by the following proposition :

Proposition 3.1. *(See ([4], p. 851)), Let $n \in \mathbb{N}$. Then we have*

$$\varphi(q) = \prod_{n=1}^{\infty} \frac{\left(1 - q^{2n}\right)^5}{\left(1 - q^n\right)^2 \left(1 - q^{4n}\right)^2}, \qquad \varphi(-q) = \prod_{n=1}^{\infty} \frac{\left(1 - q^n\right)^2}{\left(1 - q^{2n}\right)},$$
$$\psi(q) = \prod_{n=1}^{\infty} \frac{\left(1 - q^{2n}\right)^2}{\left(1 - q^n\right)}, \qquad \psi(-q) = \prod_{n=1}^{\infty} \frac{\left(1 - q^n\right)\left(1 - q^{4n}\right)}{\left(1 - q^{2n}\right)}.$$

And in ([4], Lemma 2.3) we can see that

$$q \prod_{n=1}^{\infty} \left(1 - q^{2n}\right)^4 \left(1 - q^{4n}\right)^4 = q\varphi^4(-q^2)\psi^4(q^2). \tag{3.5}$$

By using the definition of the two-dimensional theta function $\Phi_{k,l,m}(q)$, that is (1.3), K. S. Williams obtained very important results as :

Proposition 3.2. *(See [1], Theorem 2.4) For $q \in \mathbb{C}$ with $|q| < 1$ we have*

(a)

$$\Phi_{1,1,4}(q) = \sum_{r,s=-\infty}^{\infty} \left(r + s\sqrt{-4}\right)^2 q^{r^2 + 4s^2} = -2A_{1,4}(q)\varphi(q)\varphi(q^4) = 2qE_4^6,$$

(b)

$$\Phi_{2,1,1}(q) = \sum_{r,s=-\infty}^{\infty} \left(r + s\sqrt{-1}\right)^4 q^{r^2 + s^2} = -2B_{1,1}(q)\varphi^2(q) = 4qE_1^4 E_2^2 E_4^4.$$

Proof of Theorem 1.2. We can rewrite $H(q)$ in (1.6) as

$$\begin{aligned}
H(q) &= 2^{12}q^3\prod_{n=1}^{\infty}\left(1+q^{2n}\right)^4\left(1-q^{4n}\right)^{16} \\
&= 2^{12}q^3\prod_{n=1}^{\infty}\left(1+q^{2n}\right)^4\cdot\frac{\left(1-q^{2n}\right)^4}{(1-q^{2n})^4}\cdot\left(1-q^{4n}\right)^{16} \\
&= 2^{12}q^3\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{16}}{(1-q^{2n})^4}\cdot\left(1-q^{4n}\right)^4 \\
&= 2^{12}q^3\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{16}}{(1-q^{2n})^8}\cdot\left(1-q^{2n}\right)^4\left(1-q^{4n}\right)^4 \\
&= 2^{12}q^2\left(\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^2}{(1-q^{2n})}\right)^8\left(q\prod_{n=1}^{\infty}\left(1-q^{2n}\right)^4\left(1-q^{4n}\right)^4\right) \\
&= 2^{12}q^2\psi^8(q^2)\cdot q\varphi^4(-q^2)\psi^4(q^2),
\end{aligned} \tag{3.6}$$

where we used Proposition 3.1 and (3.5) for the last line. Now, by (1.4) and Proposition 3.2 (b), we note that

$$B_{1,1}(q) = 2\xi_3(-q)$$

and so

$$\Phi_{2,1,1}(q) = -2B_{1,1}(q)\varphi^2(q) = -4\xi_3(-q)\varphi^2(q). \tag{3.7}$$

Therefore, by (1.2) and (3.7), Eq. (3.6) becomes

$$\begin{aligned}
H(q) &= 2^{12}\xi_3(q^2)\cdot q\varphi^4(-q^2)\psi^4(q^2) \\
&= 2^{12}\left(-\frac{\Phi_{2,1,1}(-q^2)}{4\varphi^2(-q^2)}\right)\cdot q\varphi^4(-q^2)\psi^4(q^2) \\
&= -2^{10}\Phi_{2,1,1}(-q^2)\cdot q\varphi^2(-q^2)\psi^4(q^2).
\end{aligned} \tag{3.8}$$

Similarly, let us investigate the last part $q\varphi^2(-q^2)\psi^4(q^2)$ in (3.8) :

$$\begin{aligned}
q\varphi^2(-q^2)\psi^4(q^2) &= q\left(\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2}{(1-q^{4n})}\right)^2\left(\prod_{n=1}^{\infty}\frac{(1-q^{4n})^2}{(1-q^{2n})}\right)^4 \\
&= q\prod_{n=1}^{\infty}\frac{(1-q^{2n})^4(1-q^{4n})^8}{(1-q^{4n})^2(1-q^{2n})^4} \\
&= q\prod_{n=1}^{\infty}(1-q^{4n})^6 = qE_4^6 = \frac{1}{2}\Phi_{1,1,4}(q),
\end{aligned} \tag{3.9}$$

where we used Proposition 3.2 (a) for the last line. So, by (3.9), Eq. (3.8) becomes

$$H(q) = -2^{10}\Phi_{2,1,1}(-q^2)\cdot\frac{1}{2}\Phi_{1,1,4}(q) = -2^9\Phi_{2,1,1}(-q^2)\Phi_{1,1,4}(q). \tag{3.10}$$

Then by using Proposition 3.2 again, we have

$$\begin{aligned}\Phi_{1,1,4}(q) &= \sum_{r,s=-\infty}^{\infty} \left(r+s\sqrt{-4}\right)^2 q^{r^2+4s^2} \\ &= \sum_{r,s=-\infty}^{\infty} \left(r^2+2\sqrt{-4}rs-4s^2\right) q^{r^2+4s^2} \\ &= \sum_{r,s=-\infty}^{\infty} \left(r^2-4s^2\right) q^{r^2+4s^2},\end{aligned} \tag{3.11}$$

since $\sum_{r=-\infty}^{\infty} r^{2k-1}q^{r^2}=0$ for $k\in\mathbb{N}$, and also,

$$\begin{aligned}&\Phi_{2,1,1}(-q^2) = \sum_{r,s=-\infty}^{\infty} \left(r+s\sqrt{-1}\right)^4 (-q^2)^{r^2+s^2} \\ &= \sum_{r,s=-\infty}^{\infty} \left(r+s\sqrt{-1}\right)^4 (-1)^{r^2+s^2} q^{2(r^2+s^2)} \\ &= \sum_{r,s=-\infty}^{\infty} \left(r^2+2\sqrt{-1}rs-s^2\right)^2 (-1)^{r^2+s^2} q^{2(r^2+s^2)} \\ &= \sum_{r,s=-\infty}^{\infty} \left(r^4+4\sqrt{-1}r^3s-4\sqrt{-1}rs^3-6r^2s^2+s^4\right) (-1)^{r^2+s^2} q^{2(r^2+s^2)} \\ &= \sum_{r,s=-\infty}^{\infty} \left(r^4-6r^2s^2+s^4\right) (-1)^{r^2+s^2} q^{2(r^2+s^2)} \\ &= \sum_{r,s=-\infty}^{\infty} \left\{\left(r^4-3r^2s^2\right)(-1)^{r^2+s^2} q^{2(r^2+s^2)} + \left(s^4-3s^2r^2\right)(-1)^{s^2+r^2} q^{2(s^2+r^2)}\right\} \\ &= 2\sum_{r,s=-\infty}^{\infty} \left(r^4-3r^2s^2\right) (-1)^{r^2+s^2} q^{2(r^2+s^2)}.\end{aligned} \tag{3.12}$$

So, by (3.11) and (3.12), we can write (3.10) as

$$\begin{aligned}&H(q) \\ &= -2^9\left(2\sum_{r,s=-\infty}^{\infty}\left(r^4-3r^2s^2\right)(-1)^{r^2+s^2}q^{2(r^2+s^2)}\right)\left(\sum_{r,s=-\infty}^{\infty}\left(r^2-4s^2\right)q^{r^2+4s^2}\right) \\ &= -2^{10}\left(\sum_{(x_1,x_2)\in\mathbb{Z}^2}\left(x_1^4-3x_1^2x_2^2\right)(-1)^{x_1^2+x_2^2}q^{2(x_1^2+x_2^2)}\right) \\ &\qquad \times\left(\sum_{(x_3,x_4)\in\mathbb{Z}^2}\left(x_3^2-4x_4^2\right)q^{x_3^2+4x_4^2}\right) \\ &= -2^{10}\sum_{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4}(-1)^{x_1^2+x_2^2}\left(x_1^4-3x_1^2x_2^2\right)\left(x_3^2-4x_4^2\right)q^{2(x_1^2+x_2^2+2x_4^2)+x_3^2} \\ &= 2^{10}\sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4 \\ 2(x_1^2+x_2^2+2x_4^2)+x_3^2=n}}(-1)^{x_1^2+x_2^2+1}\left(x_1^4-3x_1^2x_2^2\right)\left(x_3^2-4x_4^2\right)q^n.\end{aligned} \tag{3.13}$$

Thus

$$H(q) = \sum_{n=1}^{\infty} h(n)q^n$$
$$= 2^{10} \sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4 \\ 2(x_1^2+x_2^2+2x_4^2)+x_3^2=n}} (-1)^{x_1^2+x_2^2+1} \left(x_1^4 - 3x_1^2x_2^2\right)\left(x_3^2 - 4x_4^2\right) q^n.$$

□

4 Some properties of $\varphi_1(q)$

First let us search the basic identities satisfied by $\varphi(q)$ and $\psi(q)$:

Proposition 4.1. *(See ([5], p. 15, 71, 72)) For $q \in \mathbb{C}$ with $|q| < 1$ we have*

(a)

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4),$$

(b)

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8),$$

(c)

$$\varphi(q)\psi(q^2) = \psi^2(q),$$

(d)

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2),$$

(e)

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2),$$

(f)

$$\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4).$$

Some results of Toh [6] construct $\varphi_1(q)$ with $\xi_1(q), \xi_3(q)$, and $\varphi(q)$.

Proposition 4.2. *(See ([1], Theorem 2.1)), ([6], p. 187) For $q \in \mathbb{C}$ with $|q| < 1$ we have*

(a)

$$\varphi_1(q) = -2\varphi(q)\xi_1(-q),$$

(b)

$$\varphi_2(q) = 2\varphi(q)\left(6\xi_1^2(-q) - \xi_3(-q)\right).$$

Using Proposition 4.2 we can obtain some properties as follows :

Lemma 4.1. *For $q \in \mathbb{C}$ with $|q| < 1$ we obtain*

(a)

$$\xi_1(q)+\xi_1(-q)=4\xi_1(q^2),$$

(b)

$$\xi_1(q)-\xi_1(-q)=2q\psi^4(q^2),$$

(c)

$$\xi_1^2(q)-\xi_1^2(-q)=8q\psi^4(q^2)\xi_1(q^2).$$

Proof. (a) By (1.1) we can observe that

$$\xi_1(q)=\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}=\sum_{\substack{n=1\\2|n}}^{\infty}\frac{nq^n}{1-q^{2n}}+\sum_{\substack{n=1\\2\nmid n}}^{\infty}\frac{nq^n}{1-q^{2n}} \tag{4.1}$$

and

$$\xi_1(-q)=\sum_{n=1}^{\infty}\frac{n(-q)^n}{1-(-q)^{2n}}=\sum_{\substack{n=1\\2|n}}^{\infty}\frac{nq^n}{1-q^{2n}}-\sum_{\substack{n=1\\2\nmid n}}^{\infty}\frac{nq^n}{1-q^{2n}}. \tag{4.2}$$

Thus combining (4.1) and (4.2) we have

$$\xi_1(q)+\xi_1(-q)=2\sum_{\substack{n=1\\2|n}}^{\infty}\frac{nq^n}{1-q^{2n}}=2\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{4n}}=4\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{4n}}=4\xi_1(q^2).$$

(b) In ([1], p. 85) we can see that

$$A_{1,2}(q)=-q\varphi^2(-q)\psi^2(q^4). \tag{4.3}$$

And by the definition of $A_{l,m}(q)$ in (1.4) we have $A_{1,2}(q)=\xi_1(-q)-2\xi_1(-q^2)$ so combining (4.3) we obtain

$$\begin{aligned}\xi_1(q)-\xi_1(-q)&=A_{1,2}(-q)-A_{1,2}(q)\\&=q\varphi^2(q)\psi^2(q^4)-\left(-q\varphi^2(-q)\psi^2(q^4)\right)\\&=q\psi^2(q^4)\left(\varphi^2(q)+\varphi^2(-q)\right).\end{aligned}$$

This leads that

$$\xi_1(q)-\xi_1(-q)=q\psi^2(q^4)\cdot2\varphi^2(q^2)=2q\left(\varphi(q^2)\psi(q^4)\right)^2=2q\psi^4(q^2), \tag{4.4}$$

where we used Proposition 4.1 (c) and (e).

(c) Multiplying Lemma 4.1 (a) and (b) we have

$$\begin{aligned}\xi_1^2(q)-\xi_1^2(-q)&=\left(\xi_1(q)+\xi_1(-q)\right)\left(\xi_1(q)-\xi_1(-q)\right)\\&=4\xi_1(q^2)\cdot2q\psi^4(q^2)=8q\psi^4(q^2)\xi_1(q^2).\end{aligned}$$

□

Proof of Lemma 1.3. (a) By Proposition 4.1 (a) and Proposition 4.2 (a) we obtain

$$\begin{aligned}
&\varphi_1(q)\xi_1(q)+\varphi_1(-q)\xi_1(-q)\\
&=-2\varphi(q)\xi_1(-q)\xi_1(q)+(-2\varphi(-q)\xi_1(q)\xi_1(-q))\\
&=-2\xi_1(q)\xi_1(-q)\left(\varphi(q)+\varphi(-q)\right)\\
&=-2\xi_1(q)\xi_1(-q)\cdot 2\varphi(q^4)\\
&=-4\xi_1(q)\xi_1(-q)\varphi(q^4).
\end{aligned}$$

(b) From Proposition 4.1 (b) and Proposition 4.2 (a) we have

$$\begin{aligned}
&\varphi_1(q)\xi_1(q)-\varphi_1(-q)\xi_1(-q)\\
&=-2\varphi(q)\xi_1(-q)\xi_1(q)-(-2\varphi(-q)\xi_1(q)\xi_1(-q))\\
&=-2\xi_1(q)\xi_1(-q)\left(\varphi(q)-\varphi(-q)\right)\\
&=-2\xi_1(q)\xi_1(-q)\cdot 4q\psi(q^8)\\
&=-8q\xi_1(q)\xi_1(-q)\psi(q^8).
\end{aligned}$$

(c) Multiply Lemma 1.3 (a) and (b), that is,

$$\begin{aligned}
&\varphi_1^2(q)\xi_1^2(q)-\varphi_1^2(-q)\xi_1^2(-q)\\
&=\left(\varphi_1(q)\xi_1(q)+\varphi_1(-q)\xi_1(-q)\right)\left(\varphi_1(q)\xi_1(q)-\varphi_1(-q)\xi_1(-q)\right)\\
&=-4\xi_1(q)\xi_1(-q)\varphi(q^4)\left(-8q\xi_1(q)\xi_1(-q)\psi(q^8)\right)\\
&=32q\xi_1^2(q)\xi_1^2(-q)\cdot\varphi(q^4)\psi(q^8)\\
&=32q\xi_1^2(q)\xi_1^2(-q)\psi^2(q^4),
\end{aligned}$$

where we used Proposition 4.1 (c) for the last line.

(d) By Proposition 4.1 (d), Proposition 4.2 (a) and Lemma 4.1 (a) we note that

$$\begin{aligned}
&\varphi_1(q)\varphi(-q)+\varphi_1(-q)\varphi(q)\\
&=-2\varphi(q)\xi_1(-q)\varphi(-q)+(-2\varphi(-q)\xi_1(q)\varphi(q))\\
&=-2\varphi(q)\varphi(-q)\left(\xi_1(q)+\xi_1(-q)\right)\\
&=-2\cdot\varphi(q)\varphi(-q)\cdot 4\xi_1(q^2)\\
&=-2\cdot\varphi^2(-q^2)\cdot 4\xi_1(q^2)\\
&=4\varphi(-q^2)\left(-2\varphi(-q^2)\xi_1(q^2)\right)\\
&=4\varphi(-q^2)\varphi_1(-q^2).
\end{aligned}$$

(e) By Proposition 4.1 (d), Proposition 4.2 (a), and (4.4) we can know that

$$\begin{aligned}
&\varphi_1(q)\varphi(-q) - \varphi_1(-q)\varphi(q) \\
&= -2\varphi(q)\xi_1(-q)\varphi(-q) - (-2\varphi(-q)\xi_1(q)\varphi(q)) \\
&= 2\varphi(q)\varphi(-q)\left(\xi_1(q) - \xi_1(-q)\right) \\
&= 2 \cdot \varphi(q)\varphi(-q) \cdot 2q\varphi^2(q^2)\psi^2(q^4) \\
&= 2 \cdot \varphi^2(-q^2) \cdot 2q\varphi^2(q^2)\psi^2(q^4) \\
&= 4q \cdot \varphi^2(-q^2)\varphi^2(q^2) \cdot \psi^2(q^4) \\
&= 4q\varphi^4(-q^4)\psi^2(q^4).
\end{aligned}$$

(f) Multiplying Lemma 1.3 (d) and (e) we have

$$\begin{aligned}
&\varphi_1^2(q)\varphi^2(-q) - \varphi_1^2(-q)\varphi^2(q) \\
&= \left(\varphi_1(q)\varphi(-q) + \varphi_1(-q)\varphi(q)\right)\left(\varphi_1(q)\varphi(-q) - \varphi_1(-q)\varphi(q)\right) \\
&= 4\varphi(-q^2)\varphi_1(-q^2) \cdot 4q\varphi^4(-q^4)\psi^2(q^4) \\
&= 16q\varphi(-q^2)\varphi^4(-q^4)\psi^2(q^4)\varphi_1(-q^2).
\end{aligned}$$

□

5 Conclusions

The purpose of this paper is to express two values

$$\begin{aligned}
G(q) &:= \sum_{n=1}^{\infty} g(n)q^n = 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32}(1-q^n)^{16}}{(1+q^{2n})^{12}}, \\
H(q) &:= \sum_{n=1}^{\infty} h(n)q^n = 2^{12} q^3 \prod_{n=1}^{\infty} \left(1+q^{2n}\right)^4 \left(1-q^{4n}\right)^{16}
\end{aligned}$$

into sums of squares as

$$\begin{aligned}
g(n) &= 8 \sum_{\substack{(x_1,x_2,\cdots,x_{11},x_{12}) \in \mathbb{Z}^{12} \\ x_1^2+x_2^2+\cdots+x_9^2+x_{10}^2+2(x_{11}^2+x_{12}^2)=n}} \left(x_{10}^2 - 2x_{12}^2\right), \\
h(n) &= 1024 \sum_{\substack{(x_1,x_2,x_3,x_4) \in \mathbb{Z}^4 \\ 2(x_1^2+x_2^2+2x_4^2)+x_3^2=n}} (-1)^{x_1^2+x_2^2+1} \left(x_1^4 - 3x_1^2x_2^2\right)\left(x_3^2 - 4x_4^2\right).
\end{aligned}$$

Competing Interests

The author declares that no competing interests exist.

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Appendix

We list first eighteen values of $g(n)$ in the Table 1

n	$g(n)$	n	$g(n)$	n	$g(n)$
1	16	7	-25728	13	233056
2	256	8	-49152	14	260096
3	1728	9	-44976	15	398976
4	6144	10	-53760	16	393216
5	10976	11	-55744	17	-301280
6	3072	12	73728	18	-523008

TABLE 1. $g(n)$ for n $(1 \leq n \leq 18)$

and $h(n)$ in the Table 2.

n	$h(n)$	n	$h(n)$	n	$h(n)$
1	0	7	-24576	13	540672
2	0	8	0	14	0
3	4096	9	-163840	15	385024
4	0	10	0	16	0
5	16384	11	-20480	17	-163840
6	0	12	0	18	0

TABLE 2. $h(n)$ for n $(1 \leq n \leq 18)$

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