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On the Closedness of the Convex Hull in a Locally Convex Space

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Short Research Article

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Abstract

The question of the closedness of the convex hull of the union of a closed convex set and a compact convex set in a locally convex space does not appear to be widely known. We show here that the answer is affirmative if and only if the closed convex set is bounded. The result is first proven for convex compact sets "of finite type" (polytopes) using an induction argument. It is then extended to arbitrary convex compact sets using the fact that such subsets in locally convex spaces admit arbitrarily small continuous displacements into polytopes.

Keywords: locally convex space; closed, convex and compact sets; convex hull
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The convex hull $conv(A)$ of a closed subset A in a topological vector space need not be closed (e.g., $conv(\{(x, 1/|x|) \in \mathbb{R}^2 : x \in \mathbb{R}, x \neq 0\})$ is not a closed subset of $\mathbb{R}^2$. Even more, the convex hull of a compact set need not be closed. Indeed, the set $K = \{x_n\}_1^\infty \cup \{0\}$ with $x_n = (0, \cdots, \frac{1}{n}, 0, \cdots)$ is a compact subset of the subspace E of $\ell^2(\mathbb{R})$ consisting of finite sequences and, given $\lambda_n > 0, \sum_1^\infty \lambda_n = 1$, the sequence of convex combinations $y_k = \frac{1}{\sum_1^k \lambda_n} \sum_1^k \lambda_n x_n$ converges to the infinite sequence $\{\frac{\lambda_n}{n}\}_1^\infty$, i.e., $conv(K)$ is not closed in E[1].

However, it is well established that the convex hull of a finite union of convex compact subsets in a locally convex space is compact (see e.g., [1,2]).

While investigating an optimization problem, we came across the natural and simple question: "is the convex hull of the union of a closed convex set X and a compact convex set K closed?". Surprisingly, in the context of non-necessarily metrizable topological linear spaces, the answer does

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[1] It is well-known that, if K is a totally bounded subset of a locally convex space E, then $\overline{conv}(K)$ is compact whenever E is quasi-complete (see e.g., [1 or 3]).

not seem to be widely known and could not be found in the literature.

If the set X is unbounded, the answer is certainly negative as the convex hull of the union of $X = \mathbb{R} \times \{0\}$ and $K = \{(0,1)\}$ is not closed in $\mathbb{R}^2$. Thus, the boundedness of X is necessary for a positive answer.

The object of this note is to show the converse; namely, that the convex hull of the union of a bounded closed convex set X and a convex compact set K in a locally convex space is always closed; (though it may not be compact, it is of course bounded).

In what follows, a locally convex space (*l.c.s.* for short) is a real topological linear space whose Hausdorff topology is determined by a family of continuous semi-norms. The closure of a set A is denoted by $\overline{A}$. Recall that $conv(A)$ is bounded for any bounded subset A of a l.c.s. E.

The first step consists in showing the result true when the compact convex set is a singleton.

Lemma 1. *If X is a non-empty bounded closed convex subset of a l.c.s. E and $u \in E \setminus X$, then $conv(X \cup \{u\})$ is closed.*

Proof. The set $conv(X \cup \{u\})$ is the image of the set-valued map $\Phi_u : X \rightrightarrows E$ defined by $\Phi_u(x) := [x, u]$, the line segment joining a given point $x \in X$ to the point u. Clearly, the values of Φ_u are non-empty compact and convex. We claim that the map Φ_u is upper-semicontinuous[2] . Indeed, let $x \in X$ be arbitrary but fixed and let V be an open subset of Y containing $\Phi_u(x)$. Since $\Phi_u(x)$ is compact and convex, there exists an open convex neighborhood of the origin U in E such that the convex set $\Phi_u(x)+U$ is contained in V. Since $x \in \Phi_u(x)$, it follows that $(x+U)\cap X \subset \Phi_u(x)+U$. But $u \in \Phi_u(x) \subset \Phi_u(x)+U$ as well. Thus, for any $x' \in (x+U)\cap X$, the line segment $\Phi_u(x') \subset \Phi_u(x)+U \subset V$. Hence, Φ_u is usc at x. Finally, x being arbitrary, Φ_u is usc on X which implies that its graph is a closed subset of $X \times E$. To conclude that $\Phi_u(X)$ is a closed set, consider a net $\{y_i\} \subset \Phi_u(X)$ converging to $y \in E$. We show that $y \in \Phi_u(X)$. By definition, $y_i = \lambda_i u + (1-\lambda_i)x_i$ with $x_i \in X, 0 \le \lambda_i \le 1$. A subnet of $\{\lambda_i\}$, again denoted $\{\lambda_i\}$, converges to some $\lambda \in [0,1]$.

If $\lambda = 1$, then $y = u \in \Phi_u(x)$ for all $x \in X$. Indeed, for a given continuous seminorm ρ on E, $\rho(y_i - u) = (1-\lambda_i)\rho(x_i - u)$. The boundedness of X implies that $\rho(x_i - u)$ is bounded, thus $\rho(y_i - u) \rightarrow_i 0$.

Suppose now that $\lambda = 0$. Then, for each i, $\rho(y - x_i) = \lambda_i \rho(u - x_i) \le \lambda_i M \rightarrow_i 0$, for some real constant $M > 0$, i.e., $x_i \rightarrow_i y \in X$ as X is closed. But $x \in \Phi_u(x)$ for all $x \in X$, in particular for $x = y \in \Phi_u(y) \subset \Phi_u(X)$.

Finally, if $0 < \lambda < 1$ then $x_i = \dfrac{y_i - \lambda_i u}{1-\lambda_i} \rightarrow_i \dfrac{y - \lambda u}{1-\lambda} = x \in X$, i.e., $y = \lambda u + (1-\lambda)x$. This completes the proof. □

The convex hull of a finite number of points in a vector space is called a *polytope.* We now replace the singleton in Lemma 1 by a polytope.

Proposition 2. *The convex hull of the union of a non-empty bounded closed convex subset X and a polytope P in a l.c.s. E is a closed set.*

[2]Given a set-valued map $S : X \rightrightarrows Y$ and a subset V of Y, the *upper inverse* of V by S is the set $S^+(V) = \{x \in X : S(x) \subset V\}$. If X, Y are topological spaces, S is said to be *upper semicontinuous* (*usc*, for short) at $x \in X$ if and only if the upper inverse $S^+(V)$ of any open subset V of Y containing $S(x)$ is open in X. The map S is usc on X if it is usc at every point of X.

Proof. The proof is by induction of the number of vertices of P. The case of a single vertex is settled by Lemma 1. Assume that the result holds true for any polytope consisting of n vertices and let $P = conv(\{u_1, \cdots, u_{n+1}\})$. Clearly, $conv(X \cup P) = conv(X \cup \{u_1, \cdots, u_{n+1}\})$, which being the smallest convex set containing $X \cup \{u_1, \cdots, u_{n+1}\}$ is a subset of $conv(conv(X \cup \{u_1, \cdots, u_n\}) \cup \{u_{n+1}\})$. A quick and simple calculation shows that any convex combination $y = \alpha z + (1-\alpha)u_{n+1}, 0 \leq \alpha \leq 1, z = \lambda_0 x + \sum_{k=1}^{n} \lambda_k u_k, x \in X, \sum_{k=0}^{n} \lambda_k = 1, \{\lambda_k\}_0^n \subset [0,1]$, can be rewritten as $\mu_0 x + \sum_{k=1}^{n+1} \mu_k u_k$ with $\mu_0 = \alpha\lambda_0, \mu_{n+1} = 1-\alpha$ and $\mu_k = \alpha\lambda_k, k = 1, \ldots, n$. Clearly, $\sum_{k=0}^{n+1} \mu_k = \alpha(\sum_{k=0}^{n} \lambda_k) + 1 - \alpha = 1$, i.e., $y \in conv(X \cup \{u_1, \cdots, u_{n+1}\})$. Thus, $conv(X \cup \{u_1, \cdots, u_{n+1}\}) = conv(conv(X \cup \{u_1, \cdots, u_n\}) \cup \{u_{n+1}\})$, the convex hull of a bounded closed convex set and a singleton, which is closed by Lemma 1. □

To prove the main result of this note, we shall make use of an approximation result for compact subsets of locally convex spaces by polytopes.

Lemma 3. *Let K be a non-empty compact subset of a l.c.s. E, and let U be a convex open symmetric neighborhood of the origin in E. Then there exists a continuous mapping $\pi_U : K \to E$ satisfying the following properties:*

(i) $\pi_U(K)$ is contained in a polytope P_U.

(ii) $y - \pi_U(y) \in U$, for every $y \in K$.

Proof. Let $N_U := \{u_1, ..., u_n\}$ be a finite subset of K such that the collection $\{(u_k + U) \cap K : k = 1, ..., n\}$ forms an open cover of K. Consider the so-called Schauder projection
$\pi_U : \bigcup_{k=1}^{n} \{u_k + U\} \longrightarrow P_U = conv(N_U)$ defined by:

$$\pi_U(y) := \frac{1}{\sum_{k=1}^{n} \mu_k(y)} \sum_{k=1}^{n} \mu_k(y) u_k, \text{ for all } y \in \bigcup_{k=1}^{n} \{u_k + U\},$$

where for $k = 1, ..., n$, $\mu_k(y) := \max\{0, 1 - \rho_U(y - u_k)\}$ and ρ_U is the Minkowski functional (a semi-norm) associated to the neighborhood U. It follows from the convexity of U that:

$$y - \pi_U(y) \in U \text{ for all } y \in \bigcup_{k=1}^{n} \{u_k + U\} \text{ and } \pi_U(y) \in P_U = conv(N_U).$$

□

We end with the main result of this note:

Theorem 4. *The convex hull of the union of a non-empty closed convex set X and a non-empty convex compact set K in a l.c.s. E is a closed set if and only if X is bounded.*

Proof. As pointed out earlier, the necessity is quite clear as for unbounded X, $conv(X \cup K)$ may not be closed. To prove the converse, let $\{U_i\}_{i \in I}$ be a filter base of convex open neighborhoods of the origin in E indexed by a directed set I with $\bigcap_{i \in I} U_i = \{0_E\}$ and, for each $i \in I$, let $P_i = P_{U_i}$ be the polytope provided by Lemma 3 and associated to U_i and K. Note that $P_i \subseteq K$ as K is a convex set. Define $F = conv(X \cup K)$ and $F_i = conv(X \cup P_i), i \in I$. By Proposition 2, for all $i \in I$, F_i is a closed convex subset of F. We show first that the Kuratowski inferior limit[3] $\liminf_i F_i$ of the family

[3] Given a directed index set $(I, \succeq)$, the *Kuratowski inferior limit* of a family of subsets $\{F_i\}_{i \in I}$ of a topological space Z is defined as $\liminf_i F_i := \{z \in Z :$ for every open neighborhood U of z in Z, there exists $i \in I$ such that $U \cap F_j \neq \emptyset$ for all $j \in I, j \succeq i\}$. The set $\liminf_i F_i$ is always a closed set. The reader is referred to [4] for details.

$\{F_i\}_{i\in I}$, contains F. Indeed, consider a point $F \ni z = \alpha x + (1-\alpha)y, x \in X, y \in K, \alpha \in [0,1]$. For any given open neighborhood U of the origin in E, there exists $i \in I$, such that $U_j \subset U$ for any $j \succeq i$ in I. For any $j \succeq i$, the points $y_j = \pi_{Uj}(y) \in P_j$ provided by Lemma 3 verify $y - y_j \in U_j$. Putting $z_j = \alpha x + (1-\alpha)y_j \in F_j$, it follows $z - z_j = (1-\alpha)(y - y_j) \in U_j \subset U$, i.e. $(z+U) \cap F_j \neq \emptyset$ for any $j \succeq i$. This shows that $F \subseteq \liminf_i F_i$. To show the reverse inclusion, note first that for any family of sets $\{F_i, G_i\}_{i\in I}$ it always holds $\liminf_i (F_i \cap G_i) \subseteq (\liminf_i F_i) \cap (\liminf_i G_i)$. Taking $G_i = F$ for all $i \in I$, we have:

$$\liminf_i F_i = \liminf_i (F_i \cap F) \subseteq (\liminf_i F_i) \cap \overline{F} \subseteq \overline{F}.$$

Hence, $\liminf_i F_i = \overline{F}$. We end the proof by showing that $\overline{F} \subseteq F$. To this end, let $\overline{F} \ni y = \lim_i y_i$, with $y_i = \lambda_i x_i + (1-\lambda_i)u_i, 0 \leq \lambda_i \leq 1, x_i \in X, u_i \in P_i \subset K$. The compactness of K and $[0,1]$ imply the existence of converging subnets (denoted again, with no loss of generality) $\lambda_i \to_i \lambda \in [0,1]$, and $u_i \to_i u \in K$. Given a continuous semi-norm ρ on E, three cases are possible.

Assume $\lambda = 1$. For each $i \in I, 0 \leq \rho(y_i - x_i) = (1-\lambda_i)\rho(u_i - x_i) \leq (1-\lambda_i)M$ for some $M > 0$ (because $K - X$ is a bounded set). Then, $0 \leq \rho(x_i - y) \leq \rho(x_i - y_i) + \rho(x_i - y_i) \to_i 0$. That is $x_i \to_i y \in X \subset F$.

In case $\lambda = 0$, for each $i \in I, 0 \leq \rho(y_i - u_i) = \lambda_i \rho(x_i - u_i) \leq \lambda_i M$. Then, $0 \leq \rho(y_i - u) \leq \rho(y_i - u_i) + \rho(u_i - u) \to_i 0$. That $y_i \to_i u = y \in K \subset F$.

Finally, if $0 < \lambda < 1$, the point $x = \dfrac{y - (1-\lambda)u}{\lambda}$ is the limit of (a subnet) $\{x_i\} \subset X$. Hence, $x \in X$ and $y = \lambda x + (1-\lambda)u \in conv(X \cup K) = F$. This ends the proof. □

Conclusion

We have established that the convex hull of the union of a closed convex set X and a compact convex set K in a locally convex space is a closed set if and only if the closed convex set X is bounded. The result does not appear to be widely known and the argument used is rather elementary. It is first established for the case where K is a finite polytope, then extended to arbitrary convex compact sets (and more generally so-called *Klee admissible sets*) by means of an approximation property. Implications of the result to optimization of semicontinuous convex functionals will be studied elsewhere.

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Competing interests

The author declares that no competing interests exist.

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