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An Error Analysis of Implicit Finite Difference Method with Mamadu-Njoseh Basis Functions for Time Fractional Telegraph Equation

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, we proposed and analyzed the error estimate of an implicit finite difference method with Mamadu-Njoseh as basis functions for time fractional telegraph equation. To enhance the efficacy of the method we first transform the Caputo type fractional derivative into Riemann-Liouville derivatives. The error analysis of the method is stated and proven. Also, the optimal results for scalars unknown in L_∞ norm were derived for the two-dimensional case. Numerical illustrations are presented to test the reliability of the analytical and computed results. The resulting numerical evidence shows that the proposed method convergences more rapidly than the standard finite difference method. MAPLE 18 is used for all mathematical procedures in this paper.

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1 Introduction

The telegraph equation (otherwise known as the transmission line model) is a coupled partial differential equation that models the flow of voltage and current on a transmission line in time and distance. The equation was designed by Oliver Heaviside in 1876 in the course of developing the transmission line model. The equation has revolved, over the years with direct applications to transmission lines involving all frequencies, such as telephone lines, radio frequency, telegraph wires, power line and wire radio antenna [1].

In view of Wang et al. [2], a typical time fractional telegraph equation is given as

$$ {}_0^C D_t^{\beta} u(x,t) + u_t(x,t) - u_{xx}(x,t) = g(x,t), x \in [0,T], t > 0, \tag{1.1} $$

with the initial conditions

$$ \left.\begin{aligned} u(x,0) &= u_0(x) \\ u_t(x,0) &= u_1(x) \end{aligned}\right\} x \in [0,T], \tag{1.2} $$

and boundary conditions

$$ \left.\begin{aligned} u(0,t) &= 0 \\ u(T,t) &= 0 \end{aligned}\right\} t > 0, x \in [0,T] \tag{1.3} $$

where $1 < \beta < 2$ and $g(x,t)$ is the source term.

In recent literature, there exist some numerical techniques for solving partial differential equations (PDEs), which include the spectral method [3], finite volume method [4], finite difference method [5], mixed finite element method [6], finite element method [7] and H^1 – Galerkin mixed finite element method [2]. Yazdani *et. al.* [8] studied the numerical solution of space fractional advection-diffusion equation adopting the finite volume-element method. The work expressed the fractional derivative composition in the Grunwald-Letnikov form. The convergence and stability of the method was also studied which resulted to the conclusion that full discretization is possible and stable in as much as the mesh graded size is sufficiently small. In like fashion, Hao et. al. [9] considered the Galerkin finite element method (GFEM) for the solution of two-sided one-dimensional diffusion equation with variable coefficients. They reformulated the governing problem into a low-ordered term that is fractional by mere introduction of an extra parameter. It was argued that the GFEM is far superior to that of the Petro-Galerkin method in the sense that the GFEM can easily be extended to three-dimensional variable coefficients.

However, the stability and convergence analysis of this method was not treated. Superconvergence of the finite element method as applied to time-fractional diffusion equation (TFDE) governed by a time-space diffusivity was studied by An [10]. Weak singularity of the model problem was studied at $t = 0$. Also, the fully discrete scheme on a bounded mesh, and fully discrete conforming finite element method was investigated. The author in conclusion remarked that superconvergence is achievable if temporary mesh pints are set at $r \geq (2-\alpha)/\alpha$, where r is graded mesh size and $0.5 < \alpha < 1$. Liu *et. al.*, [6] also studied the numerical solution of time-fractional partial differential equations. The mixed element method (MEM) was adopted as the numerical solver of the problem. The work of [6] is quiet fascinating in the sense that the Caputo fractional derivative was discretized in time via the two-step method (otherwise, finite difference method), and spatial direction was discretized using the mixed finite element method.

There are few works existing in literature on the numerical methods of the telegraph equation. For instance, Wang *et. al.* [2] applied the H^1 – Galerkin mixed finite element method (H^1-GMFEM) for the solution of time fractional telegraph equation. In line with the approach of Liu et. al. [6], the authors also fully discretized in time the Caputo fractional derivative using the finite difference method, and discretized in space using the H^1-GMFEM. For more on this, see Wei *et. al.* [11], and Zhao and Li [7].

Mamadu-Njoseh polynomials (MNP) are orthogonal polynomials developed by Njoseh and Mamadu [12] for seeking the approximate solution of many linear and nonlinear problems in the field of applied mathematics. The polynomials were constructed in the interval [-1,1] with respect to the weight function $w(x) = 1 + x^2$. These polynomials so far has contributed so much in seeking the approximate solutions of integral equations, boundary value problems, singular initial value problems in ordinary differential equation, integro-differential equation, delay differential equations. However, these polynomials are implemented through an appropriate numerical scheme as basis functions. For instance, the MNP were adopted by Ahmed and Singh [13] for the solution of integral equation via the Galerkin method. In like manner, Al-Humedi and KadhimMunaty [14] studied comparatively the MNP alongside Chebyshev and Laguerre polynomials for the solution of first kind intergral equation by the spectra petro-Galerkin method. Montazer et al. [15] studied the MNP alongside non-uniform Haar wavelets for the numerical treatment of linear Volterra integral equations.

Problems involving fractional order have equally solved applying these polynomials as seen in the literature Xie [16]. Njoseh and Mamadu [17] applied the MNP as trial functions for the solution of fifth order boundary value problems via the power series approximation method. These polynomials were also used by Mamadu and Njoseh [18] for the solution of Votterra integral equation via the Galerkin Method. Mamadu and Njoseh [19] considered the Mamadu-Njoseh polynomials in orthogonal collocation methods for the solution of integro-differential equations. Ogeh and Njoseh [20] constructed a modified variational iteration method for the solution of fifth and sixth order boundary value problems adopting the Mamadu-Njoseh polynomials as trial functions. In like manner, Njoseh and Musa [21] adopted these polynomials for the solution of pantograph-type delay differential equation in a variational iteration approach. Also, Mamadu and Ojarikre [22] proposed a reconstructed Elzaki transform method (RETM) for the solution of delay differential equation using Mamadu-Njoseh polynomials as basis functions. A perturbation by decomposition technique was considered by Mamadu and Tsetimi [23] adopting the MNP as basis functions for the solution of singular initial value problems.

This paper will centre on Mamadu-Njoseh polynomials as basis functions in a discretization scheme. Thus, a novel finite diference method with Mamadu-Njoseh basis functions for the time-fractional telegraph equation (1.1) will be proposed. An optimal error analysis for scalar unknowns in L_∞ norm will be established for the equation (1.1).

2 Preliminaries

Let equation (1.1) can be transformed into a fractional differential equation with the Riemann – Lionville derivatives [24]

$$ {}_0^R D_t^\beta [u - u_0](x,t) + u_t\,(x,t) - u_{xx}\,(x,t) = g(x,t). \tag{1.4} $$

By definition,

$$ {}_0^R D_t^\beta\,(u_0) = \frac{d}{dt}\frac{1}{\gamma(1-\beta)}\int_0^t (t-s)^{-\beta} u_0\,ds = \frac{u_0}{\gamma(1-\beta)}\frac{d}{dt}\left(\frac{1}{1-\beta}\,t^{1-\beta}\right) = \frac{u_0}{\gamma(1-\beta)}\,t^{-\beta}\,. $$

Thus,

$$ {}_0^R D_t^\beta\,u\,(x,t) = \frac{1}{\gamma(-\beta)}\int_0^t (t-s)^{-\beta-1}\,u(s)\,ds. \tag{1.5} $$

Let [0, 1] be partitioned as $0 = t_0 < t_1 < \cdots < t_n = 1$. by using $nt_j = j, j = 1,2, \dots, n$, we approximate (1.4) in time step as

$$ {}_0^R D_t^\beta\,u(x,t) = \frac{1}{\gamma(-\beta)}\int_0^{t_j}\left(t_j - s\right)^{\beta+1} u(s)\,ds. $$

let $t_j = t_j w + s$, we obtain,

$$ {}_0^R D_t^{\beta}\, u(x,t_j) = \frac{t_j{}^{-\beta}}{\gamma(-\beta)} \int_0^1 \frac{u(t_j - t_j w) - u(0)}{w^{\beta+1}}\, dw = \frac{t_j{}^{-\beta}}{\gamma(-\beta)} \int_0^1 F(w) w^{-\beta-1}\, dw, \quad (1.6) $$

where $F(w) = u\big(t_j - t_j w\big) - u(0)$.

Using the quadrature formula [24], the integral is replaced with $t_j = {}^{n}/_{j}, n = 0$, for each j, to obtain

$$ {}_0^R D_t^{\beta}\, u\,(x,t) = \frac{1}{\gamma(-\beta)} \left[\sum_{r=0}^{j} \beta_{rj}\ u\big(t_j - t_r\big) + G_j(g)\right], $$

where,

$$ \left\|G_j(g)\right\| \le k_j{}^{\beta-2}\, sup_{0\le t\le T} \left\|u''\big(t_j - t_j t\big)\right\|. $$

Thus,

$$ {}_0^R D_t^{\beta}\, u\,(x,t_j) = \frac{\Delta t^{-\beta}}{r(2-\beta)} \sum_{r=0}^{j} (-\beta)(1-\beta) j^{-\beta} \beta_{rj}\, u\big(t_j - t_r\big) + \frac{t_j{}^{-\beta}}{\gamma(-\beta)} G_j(g) $$

$$ = \Delta t^{-\beta} \sum_{r=0}^{j} \frac{(-\beta)(1-\beta) j^{-\beta} \beta_{rj}}{r(2-\beta)} u\big(t_j - t_r\big) + \frac{t_j{}^{-\beta}}{\gamma(-\beta)} G_j(g) $$

$$ = \Delta t^{-\beta} \sum_{r=0}^{j} w_{rj}\, u\big(t_j - t_r\big) + \frac{t_j{}^{-\beta}}{\gamma(-\beta)} G_j(g), $$

Where,

$$ \gamma(2-\beta) w_{rj} = (-\beta + \beta^2) j^{-\beta} \beta_{rj}, $$

such that w_{rj} and β_{rj} satisfies

$$ w_{rj} = \frac{1}{\gamma(2-\beta)} \begin{cases} 1, & r = 0 \\ -2r^{1-\beta} + (r-1)^{1-\beta} + (r+1)^{1-\beta}, & r = 1,2,\dots,j \\ -(\beta-1) r^{-\beta} + (r-1)^{1-\beta} - r^{1-\beta}, & r = j \end{cases}, $$

$$ \beta_{rj} = \frac{1}{\beta(1-\beta) j^{-\beta}} \begin{cases} -1, & r = 0 \\ -2r^{1-\beta} - (r-1)^{1-\beta} - (r+1)^{1-\beta}, & r = 1,2,\dots,j \\ (\beta-1) r^{-\beta} - (r-1)^{1-\beta} + r^{1-\beta}, & r = j \end{cases}. $$

2.1 Discretization in time

Let consider the finite difference method [25] of (1.4) at $t = t_j$, we get

$$ {}_0^R D_t^{\beta} [u(x,t) - u_0(x,t)]|_{t=t_j} + u_t\big(x,t_j\big) = g\big(x,t_j\big) + u_{xx}(x,t_j) \quad (1.7) $$

$$ \Longrightarrow \quad \Delta t^{-\beta} \sum_{r=0}^{j} w_{rj} \big[u\big(t_j - t_r\big) - u(0)\big] + \frac{t_j{}^{-\beta}}{\Gamma(-\beta)} G_j(g) + u_t\big(x,t_j\big) = g\big(x,t_j\big) + u_{xx}\big(x,t_j\big), $$

or

$$ {}_0^R D_t^{\beta}[u(x,t)-u_0(x,t)]|_{t=t_j} = \Delta t^{-\beta}\sum_{r=0}^{j} w_{rj}[u(t_j-t_r)-\mathrm{u}(0)] + \frac{t_j{}^{-\beta}}{\Gamma(-\beta)}G_j(\mathrm{g}) \,. \qquad (1.8) $$

Denote $u_j \approx u(x,t_j)$, we obtain,

$$ \Delta t^{-\beta}\sum_{r=0}^{j} w_{rj}\left[u_{j-r}-u_0\right] + \frac{u_j-u_{j-1}}{\Delta t} = G(x,t_j) + u_{xx}(x,t_j). $$

Let $r=0$, we obtain,

$$ \Delta t^{-\beta}\left(w_{oj}u_j - w_{oj}u_o\right) + \Delta t^{-\beta}\sum_{r=1}^{j} w_{rj}\left[u_j - r\right] + \Delta t^{-\beta}\sum_{r=0}^{j} w_{rj}u_0 + \frac{u_j-u_{j-1}}{\Delta t} = G(x,t_j) + \frac{u_{j+1}-2u_j+u_{j-1}}{\Delta t^2} $$

$$ \Rightarrow (u_j-u_0)w_{0j} + \sum_{r=1}^{j} w_{rj}u_{j-r} + \sum_{r=0}^{j} w_{rj}u_0 + + \Delta t^{-\beta}\left[\frac{u_j-u_{j-1}}{\Delta t} - \frac{u_{j+1}+2u_j-u_{j-1}}{\Delta x^2}\right] = g(x,t_j). $$

But from, we find [30]

$$ \sum_{r=0}^{j} w_{rj} = \frac{-\beta(1-\beta)j^{-\beta}}{(2-\beta)}\left(\frac{-1}{\beta}\right) = j^{-\beta}h_\beta, $$

where $h_\beta = \frac{1}{\Gamma(1-\beta)}$.

Thus, the implicit difference formula for (1.4) is given as

$$ (u_j-u_0)w_0,j + \sum_{r=1}^{j} w_{rj}u_{j-r} + j^{-\beta}h_\beta u_0 + \Delta t^{\beta}\left(\frac{u_j-u_{j-1}}{\Delta t} - \frac{u_{j+1}+2u_j-u_{j-1}}{\Delta x^2}\right) = g(x,t_j). \qquad (1.9) $$

2.2 Discretization in time with Mamadu-Njoseh polynomials

Suppose that an approximation to (1.4) is declined by

$$ u_j(x.t) \cong u(x,t) = \sum_{r=0}^{j} a_r(t)\varphi_r(x), \qquad (1.10) $$

where,

$a_r(t), r = 0(1)j,$ are unknown constants to be estimated, and $\varphi_r(x), r = 0(1)j,$ are Mamadu − Njoreh basis functions.

Using (1.10) on (1.9), we obtain a new implicit scheme given by

$$ \sum_{r=0}^{j} a_r(t)\varphi_r(x)w_{0,j} + \frac{j^{-\beta}}{\Gamma(1-\beta)}a_0(t) + \sum_{r=1}^{j} w_{rj}\left(\sum_{r=0}^{j-r} a_r(t)\varphi_r(x)\right) + $$

$$ \frac{\Delta t^{\beta}}{\Delta t}\sum_{r=0}^{j} a_r(t)\varphi_r(x) - \frac{\Delta t^{\beta}}{\Delta t}\sum_{r=0}^{j-1} a_r(t)\varphi_r(x) - \frac{\Delta t^{\beta}}{\Delta t}\sum_{r=0}^{jh} a_r(t)\varphi_r(x) + \frac{2\Delta t^{\beta}}{\Delta x^2}\sum_{r=0}^{j} a_r(t)\varphi_r(x) + $$

∆*tβ*∆*x2r=0j*− *1artφrx=g(x,t)* (1.11)

Equation (1.11) is collocated orthogonally ([26]-[28]) for any $j > r$ to obtain a system of $(j+1)$ linear equations which can be solve for the unknowns $a_r(t), r = 0(1)j,$ via a suitable mathematical software with estimate $\beta, \Delta t, \Delta t^2, w_{rj}$ clearly given. The approximate solution to (1.4) is obtain by substituting the known estimate into (1.10).

3 Main Results

In this section, we carry out a precise analysis of error estimate on the derived implicit formula (1.11).

Define $\beta = u_t(x,t) - u_{xx}(x,t), \Delta(\beta) = \{u: u(0) = u(T) = 0, u^{'}, u^{''} \in L_2(0,T)\}$, then the equation (1.4) can be reformulated in abstract sense as

$$ {}_0^R D_t^{\beta}[u(t) - u_0(t)]|_{t=t_j} + Bu(t) = g(t), 0 \le x \le T, t > 0, \quad (1.12)$$

Using compound quadratic formula on (1.12), we have

$$\frac{t^{-\beta}}{\Gamma(-\beta)} \sum_{r=0}^{j} \beta_{rj} \left[u\left(t_j - t_r\right) - u_0\right] + G_j(g) + Bu\left(t_j\right) = g(t_j) \quad (1.13)$$

Evaluating (1.13) $at\ r = 0$ yields

$$[\beta_{0j} + {t_j}^{\beta}\Gamma(-\beta)\beta]u(t_j) = {t_j}^{\beta}\Gamma(-\beta)g_j - \sum_{r=1}^{j} \beta_{rj}(t_j - t_r) + \sum_{r=0}^{j} \beta_{rj} u_0 - G_j(g) \quad (1.14)$$

$$\Longrightarrow \ [\beta_{0j} + {t_j}^{\beta}\Gamma(-\beta)\beta]u(t_j) = {t_j}^{\beta}\Gamma(-\beta) - \sum_{r=0}^{j} \beta_{rj} u_{jor} + \sum_{r=0}^{j} \beta_{rj} u_0, \quad (1.15)$$

Where,

$$u_j \approx u\left(t_j\right) = \sum_{r=0}^{j} a_r(t)\, u_r(x)\ .$$

Let $e_j = u - u_i$ be the error function, where u is the analytic solution and u_j, then (1.15) can also reformulated in reference to (1.11) as

$$\left[\beta_{0j} + {t_j}^{\beta}\Gamma(-\beta)\left(\frac{u_j - u_{j-1}}{\Delta t} - \frac{u_{j+1} + 2u_j - u_{j-1}}{\Delta x^2}\right)\right] u_j = {t_j}^{\beta}\Gamma(-\beta) - \sum_{r=0}^{j} \beta_{rj} u_{j-r} + \sum_{r=0}^{j} \beta_{rj} u_0. \quad (1.16)$$

Lemma 1 [24]: For $\beta \in (1,2)$, let $\{h_j\}_{j=1}^{\infty}$ with $h_1 = 1$, then

$$1 \le h_j \le \frac{sin\pi\beta}{\pi\beta(1-\beta)} j^{\beta}, j = 1(2)\infty.$$

Lemma 2 : Suppose $p_0 = 1, p_j = \beta(1-\beta)j^{-\beta} \sum_{r=1}^{j} \beta_{rj} p_{j-r}, j = 1(2)\infty,\ t$hen, $p_j \le 1$.

Theorem 1: Let U_j and u_j denotes the solution of (1.4) with the prescribed conditions, then

$$e_j \le k\Delta t^{2-\beta} + e_0,$$

where $e_0 = ||u_0 - u||$.

Proof: Let subtract (1.15) from (1.13) to obtain the error equation as

$$\left(\beta_{oj} + {t_j}^{\beta}\Gamma(-\beta)B\right)ej = -\left(\sum_{r=0}^{j} \beta_{rj} e_{j-r} + G_j(g)\right) \quad (1.17)$$

$$\Longrightarrow e_j = \left(-\beta_{oj} - {t_j}^{\beta}\Gamma(-\beta)B\right)^{-1}\left(\sum_{r=0}^{j} \beta_{rj} e_{j-r} + G_j(g)\right).$$

By definition of L_∞ norm, we obtain

$$\left\|e_j\right\| \le \left\|\left(-\beta_{oj} - {t_j}^{\beta}\Gamma(-\beta)B\right)^{-1}\right\| \left(\sum_{r=0}^{j} \beta_{rj}\left\|e_{j-r}\right\| + \left\|G_j(g)\right\|\right), \quad (1.18)$$

where B is symmetric and positive definite operator [29]. Now for any function $f(x)$, we have via spectral method that $\|f(B)\| = sup_{\tau>0} |f(\tau)|$.

Hence, using the definition of β_{rj} and (1.18), we have

$$\left\|\left(-\beta_{oj} - t_j{}^{\beta}\Gamma(-\beta)B\right)^{-1}\right\| = \left\|\left(\frac{1}{\beta(1-\beta)j^{-\beta}} - t_j{}^{\beta}\Gamma(-\beta)B\right)^{-1}\right\| \tag{1.19}$$

$$= \left\|\beta(1-\beta)j^{-\beta}\left(1-\beta(1-\beta)j^{-\beta}t_j{}^{\beta}\Gamma(-\beta)B\right)^{-1}\right\|$$

$$= \beta(1-\beta)j^{-\beta} sup_{\tau>0}\left(1-\beta(1-\beta)j^{-\beta}t_j{}^{\beta}\Gamma(-\beta)\tau\right)^{-1}.$$

By definition of gamma function, it is obvious $\Gamma(-\beta) > 0$. This implies

$$sup_{\tau>0}\left(1-\beta(1-\beta)j^{-\beta}t_j{}^{\beta}\Gamma(-\beta)\tau\right)^{-1} \leq 1.$$

Therefore,

$$\left\|\left(-\beta_{oj} - t_j{}^{\beta}\Gamma(-\beta)B\right)^{-1}\right\| \leq \beta(1-\beta)j^{-\beta} .$$

Thus, (1.18) becomes

$$\|e_j\| \leq \beta(1-\beta)j^{-\beta}\left[\textstyle\sum_{r=0}^{j}\beta_{rj}\|e_{j-r}\| + Kj^{\beta}m^{-2}sup_{0\leq t\leq T}\left\|u_{tt}{}''(t_j - t_r)\right\|\right], \tag{1.20}$$

$\Delta t = 1, t_m = m.$

Equation (1.20) can be reformulated into the form,

$$\|e_j\| \leq b + \beta(1-\beta)j^{-\beta}\textstyle\sum_{r=0}^{j}\beta_{rj}\|e_{j-r}\|, \tag{1.21}$$

where, $b = \beta(1-\beta)_j{}^{-\beta}Km^{-2}\|u''\|_{L_\infty}$.

Let $j = n$, then (1.21) becomes

$$\|e_n\| \leq b + \beta(1-\beta)n^{-\beta}\left[\sum_{r=1}^{n-1}\beta_{rn}\|e_{n-r}\| + \beta_{nn}\|e_0\|\right]$$

$$\leq b + \beta(1-\beta)n^{-\beta}\left[\textstyle\sum_{r=1}^{n-1}\beta_{rn}\left(bh_{n-r} + p_{n-r}\left||e_0|\right|\right) + \beta_{nn}\|e_0\|\right]$$

$$= ah_n + p_n\|e_0\|$$

where ,

$$h_n = 1 + \beta(1-\beta)n^{-\beta}\sum_{r=1}^{n-1}\beta_{rn}h_{n-r}, n = 2,3,\ldots,$$

$$p_n = \beta(1-\beta)n^{-\beta}\sum_{r=1}^{n}\beta_{rn}p_{n-r}, n = 1(2)\beta, p_0 = 1.$$

Now using lemma 1 and lemma 2, we have from (1.21),

$$\|e_j\| \leq bh_j + p_j\|e_0\| \leq \beta(1-\beta)Km^{-2}\|u''\|_{L_\infty}.h_j + p_j\|e_0\|$$

$$\leq \beta(1-\beta)Km^{-2}\|u''\|_{L_\infty}\frac{\sin\pi\beta}{\pi\beta(1-\beta)}j^{\beta} + \|e_0\|$$

$$\leq 1$$

$$\leq k\Delta t^{2-\beta} + \|e_0\|.$$

4 Numerical Illustration

To test the reliability and accuracy of the proposed method, we consider the example below:

$$ {}_{0}^{C}D_t^{\beta}u(x,t) + u_t(x,t) - u_{xx}(x,t) = 2(x^2-x)t\left(\frac{\Gamma(3-\beta)+t^{1-\beta}}{\Gamma(3-\beta)}\right) - 2t^2, x \in [0,1],\ \ t \in (0,1], \quad (1.22)$$

with the initial conditions

$$\left.\begin{array}{l} u(0,0) = 0 \\ u_t(x,0) = 0 \end{array}\right\},$$

and boundary conditions

$$\left.\begin{array}{l} u(0,t) = 0 \\ u(1,t) = 0 \end{array}\right\}, t > 0.$$

Applying the scheme (1.11) on (1.22) at $j = 3$ with parameters $\beta = 1.5$, $\Delta x = 1/64$, $\Delta t = 1/1000$ at $t = 0$, and values of $w_{rj}, r = 0(1)3$, estimated as $w_{0,3} = 1$, $w_{1,3} = -0.7294368868$, $w_{2,3} = 0.09204003089$ and $w_{3,3} = 0.1817856084$. Results are presented Table 1 and Fig. 1 using MAPLE 18.

Table 1. Maximum error at $\beta = 1.5$

j	L_∞ Error (Proposed method)	L_∞ Error [11]
20	5.6445E-008	9.877022E-003
40	3.5198E-006	3.477002E-003
80	1.5441E-005	1.232302E-003
160	6.3327E-005	4.249358E-004

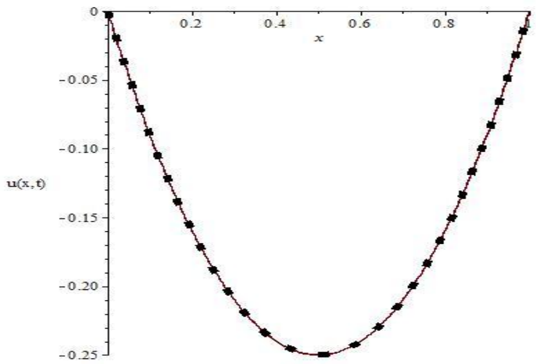

Fig. 1. Comparison of computed solutions and exact solutions

Similarly at $j = 3$ with parameters $\beta = 1.8$, $\Delta x = 1/32$, $\Delta t = 1/1000$ at $t = 0$, and values of $w_{rj}, r = 0(1)3$, estimated as $w_{0,3} = 1$, $w_{1,3} = -0.3105422252$, $w_{2,3} = 0.05806019726$ and $w_{3,3} = 0.06480727693$. Results are presented in Table 2 and Fig. 2 using MAPLE 18.

Table 2. Maximum error at $\beta = 1.5$

j	L_∞ Error (Proposed method)	L_∞ Error [11]
20	6.6376E-004	1.1484477E-002
40	1.4188E-006	5.148878E-003
80	2.4280E-007	1.998394E-003
160	3.9604E-008	8.980387E-004

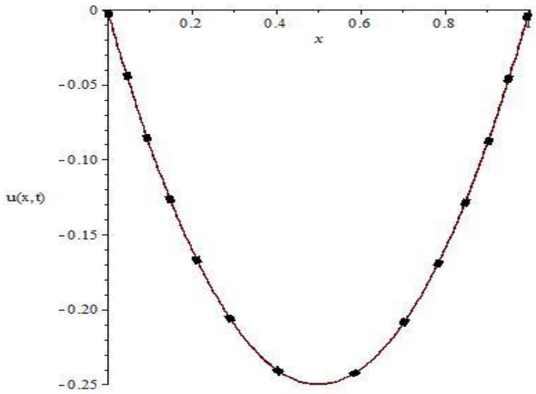

Fig. 2. Comparison of computed solutions and exact solutions

5 Discussion of Results

Numerical evidences to test the reliability and accuracy of the proposed method are presented in tables and figures. Tables 1 and 2 shows the maximum error of the proposed method at $j = 20,40,80,$ and 160. In comparison with the finite difference method in [30-32] show the superiority of the proposed method with maximum errors of order 10^{-8} and 10^{-8}, respectively. Results are also presented in graphs showing the comparison of results as shown in the Figs. 1 and 2.

6 Conclusion

In this paper, we have successively proposed an implicit finite difference method with Mamadu-Njoseh polynomials as basis functions for the time fractional telegraph equation. Numerical illustration of the proposed method showed convergence and accuracy than the standard finite difference method. The optimal error analysis of the proposed method was investigated in L_∞ norm for two dimensional case. The result showed that the method is of order $(2 - \beta)$.

Competing Interests

Authors have declared that no competing interests exist.

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