



On the Stability Analysis of Linear Unperturbed Non-integer Differential Systems

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, the stability of non-integer differential system is studied using Riemann-Liouville and Caputo derivatives. The stability notion for determining the stability/asymptotic stability or otherwise fractional differential system is given. Example is provided to demonstrate the effectiveness of the result.

Keywords: Stability; asymptotic stability; riemann-liouville derivative; caputo derivative; fractional differential systems.

1 Introduction

Calculus as created by Isaac Newton, a British scientist as well as Gottfried Leibniz, a self taught German mathematician in the 17th century and known in the early days as infinitesimal calculus is a mathematical discipline focused on limits, continuity derivatives, integrals and infinite series. There are broadly two forms of calculus namely, the integer-order calculus and the non-integer order calculus.

It is customary to discuss differential calculus before discussing integral calculus in integer-order calculus. In non-integer calculus, it is necessary to deal with integral calculus before the differential calculus. This is

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because differential calculus is defined in terms of the integral calculus. There are two fundamental distinctions between fractional order derivative and integer-order derivative. The non-integer derivative is concerned with the whole time domain for a mechanical or physical process, while the integer-order derivative indicates a variation or certain attribute at particular time. Also, the fractional order derivative is related to the whole space for a physical process, while the integer-order derivative describes the local properties of certain position. This is the reason that many real world physical systems are characterized by the fractional order state equations [1-6]. Fractional differential equations are generalization of classical integer-order differential equations through the application of fractional calculus [7].

For more than three centuries, fractional calculus has received increasing attention due to its applications in many fields such as: synchronization, tracking controller, physics, control engineering, signal processing and complex systems. Continuing technological developments have required new methods in basic sciences, especially in mathematics for analysis and design of physical systems and their control tools. These methods which are easily implemented with the advancement of high speed computers facilitate better characterization, design tools and control performance of modern technological products of engineering systems of developing civilization.

These developments which had covered only static systems models involving geometry and algebra until 1965, had started using dynamical models involving differential and integral calculus and now have been accelerating since 1960's with the fractional order systems have gained momentum with high speed computers. Hence, FDE's have become a powerful tool in studying, designing and control of engineering products of the present world and it still constitutes a popular research area resulting with new definitions of fractional derivative and its applications in need.

Stability is one of the most important objects in the analysis and design of dynamical systems. If the differential equation of a system is not stable, the system may burn out, disintegrate or saturate when a signal is applied [8]. Therefore, an unstable system is useless in practice and needs a stabilization process via an additional control elements [9]. Stability analysis is a central task in the study of fractional differential systems and fractional control [10,11]. In fact, many systems in real world are now better characterized by FDE's and analysed by numerical techniques developed for solving differential equations involving non-integer derivatives. Recently, due to evolving interest in the study of stability of systems occasioned by the importance of providing and ensuring the stability of differential systems, much work has been done in this direction.

In this work, the stability of linear fractional differential equations is studied using Caputo and Riemann-Liouville derivatives. The stability notion for linear fractional differential equations is presented. Example is given to show the applicability of the stability notion. In the remainder, the preliminaries and definitions are given in section 2 while the analysis and conclusion are given in sections 3 and 4 respectively.

2 Preliminaries and Definitions

Extraordinary differential equation as fractional differential equation is sometimes called is an evolving concept. It is therefore not unusual to define some concepts and functions in terms of other definitions and terms. In this section, some definitions and concepts that will be used in the paper are given.

Definition 2.1 (Gamma Function): The gamma function represented by Γ (the capital letter gamma from the Greek alphabet) is one commonly used extension of the factorial function to complex numbers. Gamma function is the generalization of the factorial function to non-integral values, introduced by the Swiss mathematician Leonhard Euler in the 18th century. For any positive integer n ,

$\Gamma(n) = (n - 1)!$. But this formula is meaningless if n is not an integer. To extend the factorial to any real number $x > 0$ (whether or not x is a whole number), the gamma function is defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad (x > 0)$$

Definition 2.2: The Riemann-Liouville derivative and the Caputo derivative will be used in the analysis.

The Riemann-Liouville derivative is defined as

$${}_{RL}D_{a,t}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^{\infty} (t-\tau)^{n-\alpha-1} x(\tau) d\tau, \quad (n-1 \leq \alpha < n)$$

And the Caputo derivative is defined as

$${}_cD_{a,t}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\infty} (t-\tau)^{n-\alpha-1} x^{(n)}(\tau) d\tau, \quad (n-1 \leq \alpha < n)$$

where $\Gamma(\cdot)$ is the Euler's integral (gamma function).

The Laplace transform of the Riemann-Liouville fractional derivative ${}_{RL}D_{a,t}^{\alpha}x(t)$ is given as

$$\int_0^{\infty} e^{-st} {}_{RL}D_{a,t}^{\alpha}x(t) dt = S^{\alpha}X(s) - \sum_{k=0}^{n-1} S^k [D^{\alpha-k-1}x(t)]_{t=a} \quad (n-1 \leq \alpha < n)$$

Similarly, the Laplace transform of the Caputo fractional differential derivative ${}_cD_{a,t}^{\alpha}x(t)$ is given as

$$\int_0^{\infty} e^{-st} {}_cD_{a,t}^{\alpha}x(t) dt = S^{\alpha}X(s) - \sum_{k=0}^{n-1} S^{\alpha-k-1} x^{(k)}(a), \quad (n-1 \leq \alpha < n)$$

Definition 2.3: The Mittag-Leffler function is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad \text{where } \operatorname{Re}(\alpha) > 0, z \in \mathbb{C}$$

The two parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad (\alpha > 0, \beta > 0)$$

The Laplace transform of the Mittag-Leffler function is given as

$$\int_0^{\infty} e^{-st} t^{a\alpha-\beta-1} E_{\alpha,\beta}^{(k)}(\pm at^{\alpha}) dt = \frac{k! S^{\alpha-\beta}}{(S^{\alpha} \mp a)^{k+1}}; (R(s) > |a|^{\frac{1}{\alpha}})$$

3 Stability Analysis

The linear fractional differential system with Riemann-Liouville derivative under consideration is

$${}_{RL}D_{a,t}^{\alpha}x(t) = Ax(t), \quad (0 < \alpha < 2) \quad (3.1)$$

with initial conditions

$${}_{RL}D_{a,t}^{\alpha-k}x(t)|_{t=a} = x_{k-1} \quad (k = 1, 2) \quad (3.2)$$

where $x(t) = (x_1(t), x_2(t), x_3(t) \dots \dots x_n(t))^T \in R^n$, $A = (a_{ij})_{n \times n} \in R^{n \times n}$

The stability of the systems is determined by the nature of the eigenvalues of A. The solution of (3.1) with (3.2) is given by

$$x(t) = (t - a)^{\alpha-1} E_{\alpha,\alpha}(A(t - a)^\alpha)x_0 + (t - a)^{\alpha-2} E_{\alpha,\alpha-1}(A(t - a)^\alpha)x_1$$

Similarly, consider the system

$${}_c D_{a,t}^\alpha x(t) = Ax(t), \quad (\alpha < t) \quad (3.3)$$

with the initial condition

$$x^{(k)}(a) = x_k \quad (k = 0,1) \quad (3.4)$$

The solution of (3.3) with (3.4) is given by

$$x(t) = x_k E_{\alpha,1-k}(At^\alpha)$$

which reduces to

$$x(t) = x_0 E_{\alpha,1}(At^\alpha)$$

since other terms are the obtainable.

To establish the stability or otherwise of the solutions, we state the following theorem

Theorem: *The autonomous fractional differential system (3.1) with (3.2) and the system (3.3) with (3.4) are asymptotically stable iff $|\arg(\text{spec}(A))| > \frac{\alpha\pi}{2}$.*

4 Discussion

For the system (3.1), the components of the state decay towards O like $t^{-\alpha-1}$. Also, the system is stable if and only if either it is asymptotically stable or those critical eigenvalues which satisfy $|\arg(\text{spec}(A))| = \frac{\alpha\pi}{2}$ have the same algebraic and geometric multiplicities. Similarly, the components of the state decay towards O like $t^{-\alpha+1}$ in system (3.3) and the system is stable if either it is asymptotically stable or those critical eigenvalues which satisfy $|\arg(\text{spec}(A))| = \frac{\alpha\pi}{2}$ have the same algebraic and geometric multiplicities. In both cases, if the critical eigenvalues are such that their algebraic multiplicities are larger than their geometric multiplicities, then the solution is unstable.

Application: Consider the system ${}_{RL}D_t^\alpha x(t) = Ax(t)$

$$\text{where } A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

The eigenvalues of A are $\lambda_1 = \frac{5+i\sqrt{15}}{2}$ and $\lambda_2 = \frac{5-i\sqrt{15}}{2}$

The system is stable when $\alpha = 0.1$ and when $\alpha = 0.5$. But the system is unstable when $\alpha = 0.9$. Accordingly, the system is asymptotically stable for all $\alpha < 0.3869$, since $|\arg(\text{spec}(A))| = 0.6591..$

5 Conclusion

Central to the study of fractional differential system is the stability analysis. This is due to its importance. In this work, the stability of non-integer differential system is studied using Riemann-Liouville and Caputo derivatives. The yardstick for determining the stability/asymptotic stability of autonomous system is given. Example is given to show the applicability of the result.

Competing Interests

Author has declared that no competing interests exist.

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