Asian Research Journal of Mathematics

17(5): 14-34, 2021; Article no.ARJOM.70120
ISSN: 2456-477X

More Connections on Valuated Binary Tree and Their Applications in Factoring Odd Integers

Xingbo Wang[1*] and Yuequan Jin[1]

[1]Department of Mechatronic Engineering, Foshan University, Foshan City, China.

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2021/v17i530297
Editor(s):
(1) Dr. Xingting Wang, Howard University, USA.
Reviewers:
(1) Wang Lina, Hainan Normal University, China.
(2) Tahmineh Azizi, Kansas State University, USA.
Complete Peer review History: https://www.sdiarticle4.com/review-history/70120

Original Research Article

Received 25 April 2021
Accepted 05 July 2021
Published 10 July 2021

Abstract

This paper continues investigating connections on a valuated binary tree. By defining three types of new connections, the paper derives several new properties for the new connections, and proves that odd integers matching to the new cases can be easily and rapidly factorized. Proofs are presented for the new properties and conclusions with detail mathematical reasoning and numerical experiments are made with Maple software to demonstrate the fast factorization by factoring big odd composite integers that are of the length from 101 to 105 decimal digits. Source codes of Maple programs are also list for readers to test the experiments.

Keywords: Integer factorization; valuated binary tree; geometric relationship; connection.

2020 Mathematics Subject Classification: 11A51, 11Y99.

1 Introduction

By means of defining parallelism, connection and penetration, paper [1] investigated geometric relationships among nodes on a valuated binary tree, and it proved several properties about the connections and the

---

**Corresponding author: Email: 153668@qq.com;*

penetrations as well as some significant corollaries for fast factorization of special kind of big odd integers. The results together with the previous results obtained in the bibliographies [2] to [7] exhibit that the valuated binary tree method is a new systematic approach to analyze odd integers.

This paper follows the study of paper [1], continues the investigation on the connections and their applications in factorization of odd composite integers. By defining three new types of the connections, the paper reasons and obtains several new properties and corollaries in analyzing the odd integers.

2 Preliminaries

The terms related with the valuated binary tree, subtree, root, node, son, father and ancestors as well as symbols used in this paper can be referred in [1]. Some cited lemmas were also seen in [1].

2.1 New lemmas

Lemma 1 ([8]). Let N be an odd integer on a tree; then N's direct ancestor that is α levels away from N is calculated by $A_N^\alpha = 1 + f_N^\alpha$ if f_N^α is even or $A_N^\alpha = f_N^\alpha$ if f_N^α is odd, where $f_N^\alpha = \left\lfloor \frac{N-1}{2^\alpha} \right\rfloor$.

Lemma 2 Let $i \ge 0$, $j \ge 0$ and $\omega \ge 0$ be integers; the equality $\left\lfloor \frac{j}{2^i} \right\rfloor = \omega$ holds for either $\left\lfloor \frac{j}{2^{i-1}} \right\rfloor = 2\omega$ or $\left\lfloor \frac{j}{2^{i-1}} \right\rfloor = 2\omega + 1$.

Proof. See the following reasoning.

$$\left\lfloor \frac{j}{2^{i-1}} \right\rfloor = 2\omega \Rightarrow j = 2^i \omega + r, 0 \le r < 2^{i-1} \Rightarrow \left\lfloor \frac{j}{2^i} \right\rfloor = \omega + \left\lfloor \frac{r}{2^i} \right\rfloor = \omega$$
$$\left\lfloor \frac{j}{2^{i-1}} \right\rfloor = 2\omega + 1 \Rightarrow j = 2^i \omega + r, 2^{i-1} \le r < 2^i \Rightarrow \left\lfloor \frac{j}{2^i} \right\rfloor = \omega + \left\lfloor \frac{r}{2^i} \right\rfloor = \omega$$

3 Connections Parallel to Borders

The concept of the connection was introduced in [1]. This section mainly studies the connections parallel to the borders of a subtree.

3.1 Three types of connections parallel to the borders

Let T_A be a valuated binary tree, X and Y be nodes of T_A; assume d_l is the distance from X (or Y) to the left border B_L^A of T_A and d_r is the distance from X (or Y) to the right border B_R^A of T_A, as depicted with Fig. 1. The connection P_L^X starting from X and formed with the nodes that are d_l away from B_L^A is called a connection parallel to B_L^A and the connection P_R^X starting from Y and formed with the nodes that are d_r away from B_R^A is called a connection parallel to B_R^A. Likewise, so are the connections P_L^Y and P_R^Y defined. Since A is an ancestor of X and Y, connections parallel to the borders of T_A are said to be type-1 connections.

There is another kind of connections that are said to be type-2 ones. Seen in Fig. 2, X and Y are two nodes on the same level of T_A. Then there is a connection, denoted by P_{YR}^X, that is starting from X and parallel to the right border of T_Y; the distance from P_{YR}^X to the right border of T_Y is the same as that from X to Y. There is also a connection, denoted by P_{XL}^Y that is starting from Y and parallel to the left border of T_X.

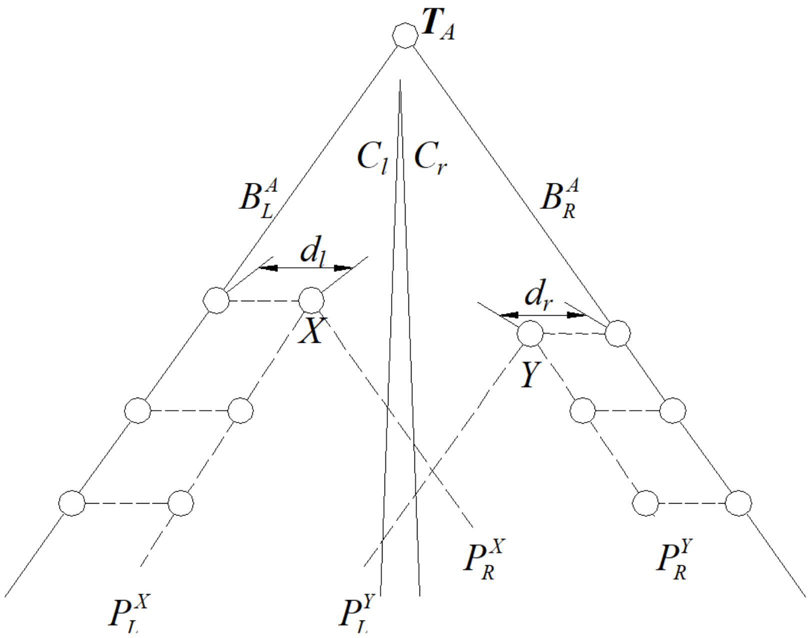

Fig. 1. Type-1 connections parallel to borders

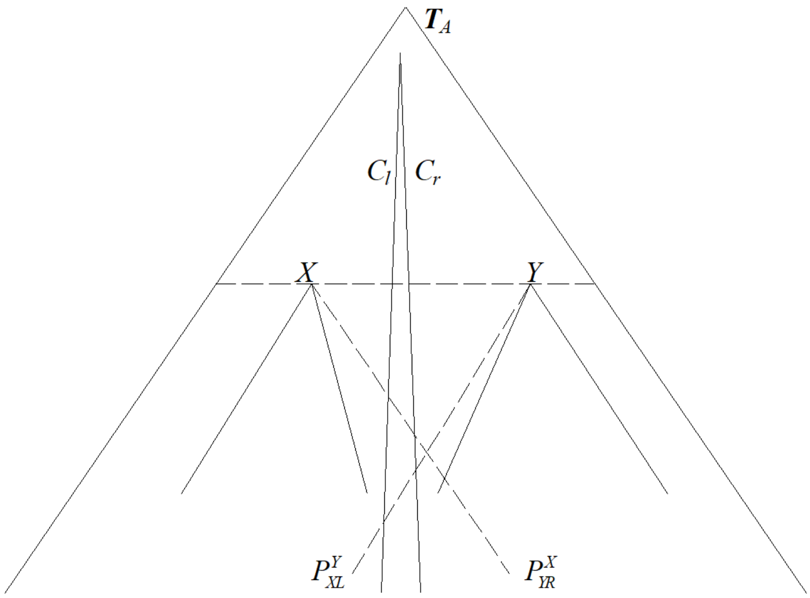

Fig. 2. Type-2 connections parallel to borders

The type-3 connections are related with two nodes X and Y that lie on different levels of T_A, as seen in Fig. 3. This kind of connections looks like the type-2 ones but they are different from the distance defined from X to the right border of T_Y or from Y to the left border of T_X. This time, the lower level is set to be a reference to calculate the distances. If $X = N_{(k,j)}^A$ and $Y = N_{(l,s)}^A$ with $k > 0$ and $l - k = \delta > 0$, then the distance d_l from Y to $N_{(\delta,0)}^X$ is defined to be the distance from Y to the left border of T_X, and the distance d_r from Y to $N_{(\delta,2^\delta-1)}^X$ is defined to be the distance from Y to the right border of T_X. Of course, some other definitions might be given if only they could simplify the calculations. For convenience, the connection is simply called a type-3 connection starting from X or Y, respectively.

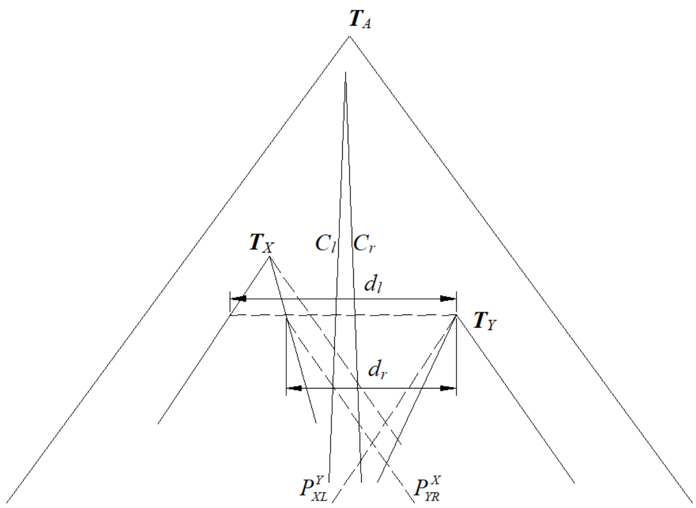

Fig. 3. Type-3 connections parallel to borders

3.2 Properties and proofs

Property 1. Let $X = N_{(k,j)}^A$ be a node of T_A. Denote P_L^X and P_R^X to be the two type-1 connections starting from X and parallel to the left and the right borders of T_A, respectively. Assume $n_i^L \in P_L^X$ and $n_i^R \in P_R^X$ are respectively the i^{th} nodes counted from X, where $i \geq 0$; then

$$n_i^L = N_{(k+i,j)}^A = 2^{k+i}(A-1) + 2j + 1$$

and

$$n_i^R = N_{(k+i,2^{k+i}-2^k+j)}^A = 2^{k+i}(A-1) + 2(2^{k+i} - 2^k + j) + 1 = 2^{k+i}(A+1) - 2(2^k - j) + 1.$$

Proof. Assume d_l and d_r are the distances from X to the left and the right borders of T_A, respectively. Since

$$N_{(k,0)}^A = 2^k(A-1) + 1,$$
$$N_{(k,2^k-1)}^A = 2^k(A-1) + 2(2^k - 1) + 1$$

and

$$N_{(k,j)}^A = 2^k(A-1) + 2j + 1,$$

it follows

$$d_r = \frac{N_{(k,2^k-1)}^A - N_{(k,j)}^A}{2} + 1 = 2^k - j,$$

and

$$d_l = \frac{N_{(k,j)}^A - N_{(k,0)}^A}{2} + 1 = j + 1.$$

Consider the case $n_i^L \in P_L^X$. Since X lies on level k of T_A, n_i^L is on level $k+i$ of T_A and it follows

$$n_i^L = N^A_{(k+i,0)} + 2(d_l - 1) = 2^{k+i}(A-1) + 2j + 1 = N^A_{(k+i,j)} .$$

Similarly, it yields

$$\begin{aligned} d_r &= \frac{N^A_{(k+i,2^{k+i}-1)} - n_i^R}{2} + 1 \Rightarrow n_i^R = N^A_{(k+i,2^{k+i}-1)} - 2(d_r - 1) = N^A_{(k+i,2^{k+i}-1)} - 2(2^k - j - 1) \\ &= 2^{k+i}(A-1) + 2(2^{k+i} - 2^k + j) + 1 = N^A_{(k+i,2^{k+i}-2^k+j)} \end{aligned}$$

Example 1. Taking in T_{21} the node 83, it is seen that, $d_l = 2$ (from 81 to 83) and $d_r = 3$ (from 83 to 87). The connection starting from 83 and parallel to the left and the right borders of T_{21} are respectively

$$P_L^{83} = \{83, 163, 323, ...\} \text{ and } P_R^{83} = \{83, 171, 347, ...\}$$

Because

$$\begin{aligned} n_0^L &= N^{21}_{(2+0,1)} = 2^2(21-1) + 2 + 1 = 83 \\ n_1^L &= N^{21}_{(2+1,1)} = 2^3(21-1) + 2 + 1 = 163 \\ n_2^L &= N^{21}_{(2+2,1)} = 2^4(21-1) + 2 + 1 = 323 \\ n_0^R &= N^{21}_{(2+0,2^2-2^2+1)} = 2^2(21-1) + 2(2^2 - 2^2 + 1) + 1 = 83 \\ n_1^R &= N^{21}_{(2+1,2^3-2^2+1)} = 2^3(21-1) + 2(2^3 - 2^2 + 1) + 1 = 171 \\ n_2^R &= N^{21}_{(2+2,2^4-2^2+1)} = 2^4(21-1) + 2(2^4 - 2^2 + 1) + 1 = 347 \end{aligned}$$

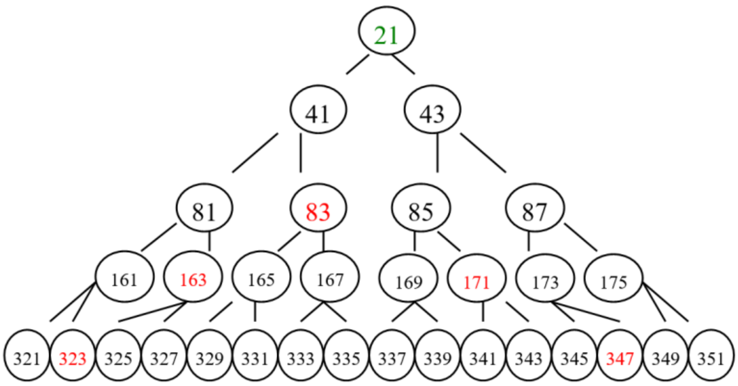

Fig. 4. Example of connections parallel to the two borders of *T*₂₁

Proposition 1. Let $X = N^A_{(k,j)}$ with $k > 0$ be a node of T_A, P_L^X and P_R^X be defined as those in Property 1; assume $n_i^L \in P_L^X$ and $n_i^R \in P_R^X$ are respectively the i^{th} nodes counted from X, where $i \geq 0$; then it holds

$$n_i^R - n_i^L = 2(2^{k+i} - 2^k) = 2^{k+1}(2^i - 1)$$

$$n_i^L = X + 2^k(2^i - 1)(A-1)$$

and

$$n_i^R = X + 2^k(2^i - 1)(A+1)$$

Proof. Direct calculation by Property 1 immediately yields

$$n_i^R - n_i^L = 2(2^{k+i} - 2^k) = 2^{k+1}(2^i - 1)$$

Now by $n_i^L = 2^{k+i}(A-1) + 2j + 1$ it follows

$$\begin{aligned}
& n_i^L = 2^{k+i}(A-1) + 2j + 1 \\
& \Rightarrow \\
& n_{i+1}^L = 2^{k+i+1}(A-1) + 2j + 1 \\
& \Rightarrow \\
& n_{i+1}^L - n_i^L = 2^{k+i}(A-1) \\
& \Rightarrow \\
& n_1^L - n_0^L = 2^{k+0}(A-1) \\
& n_2^L - n_1^L = 2^{k+1}(A-1) \\
& \ldots\ldots \\
& n_s^L - n_{s-1}^L = 2^{k+s-1}(A-1) \\
& \Rightarrow \\
& n_s^L = n_0^L + 2^k(2^s - 1)(A-1)
\end{aligned}$$

Since $n_0^L = 2^k(A-1) + 2j + 1 = N_{(k,j)}^A = X$, it yields

$$n_s^L = X + 2^s(2^s - 1)(A-1), s \ge 0$$

Likewise, it yields

$$n_s^R = X + 2^k(2^s - 1)(A+1), s \ge 0$$

Remark 1. Proposition 1 shows that, the distance from $n_i^L \in P_L^X$ to $n_i^R \in P_R^X$ is

$$d_i = \frac{n_i^R - n_i^L}{2} + 1 = 2^{k+i} - 2^k + 1$$

This is a quantity that merely depends on the level where *X* lies and the level where n_i^L lies.

Proposition 2. Let $X = N_{(k,0)}^A$ with $k > 0$ be a node on the left border of T_A; denote $T_{X_0 = X}, T_{X_1}, \ldots, T_{X_{2^k - 1}}$ to be the 2^k subtrees whose roots are respectively the 2^k nodes on level *k* of T_A. Assume $n_i^R \in P_R^X$ is the i^{th} node counted from *X*, where $i \ge 0$ and P_R^X is as defined in Property 1; then

$$n_i^R = N_{(k+i, 2^{k+i} - 2^k)}^A \in \begin{cases} T_{X_{2^k - 2^{k-i}}}, 0 \le i \le k \\ T_{X_{2^k - 1}}, i > k \end{cases}$$

Proof. Taking $j = 0$ in Property 1 immediately yields

$$n_i^R = N_{(k+i, 2^{k+i} - 2^k)}^A$$

Now consider the ancestor of $N_{(k+i, 2^{k+i} - 2^k)}^A$. When $0 \le i \le k$ direct calculation shows

$$\left\lfloor \frac{N^{A}_{(k+i,2^{k+i}-2^{k})}-1}{2^{i}} \right\rfloor = \left\lfloor \frac{2^{k+i}(A-1)+2(2^{k+i}-2^{k})}{2^{i}} \right\rfloor = 2^{k}(A-1)+2(2^{k}-2^{k-i})$$

and when $i > k$ it follows

$$\left\lfloor \frac{N^{A}_{(k+i,2^{k+i}-2^{k})}-1}{2^{i}} \right\rfloor = \left\lfloor 2^{k}(A-1)+2(2^{k}-1)+2-\frac{1}{2^{i-k-1}} \right\rfloor = 2^{k}(A-1)+2(2^{k}-1)+1$$

By Lemma 1, on level k of T_A, the ancestor of $N^{A}_{(k+i,2^{k+i}-2^{k})}$ is $N^{A}_{(k,2^{k}-2^{k-i})}$ when $0 \le i \le k$, whereas it is $N^{A}_{(k,2^{k}-1)}$ when $i > k$.

Proposition 2*. Let $X = N^{A}_{(k,2^{k}-1)}$ with $k > 0$ be a node on the right border of T_A; denote $T_{X_0}, T_{X_1}, ..., T_{X_{2^k-1}=X}$ to be the 2^k subtrees whose roots are respectively the 2^k nodes on level k of T_A. Assume $n_i^L \in P_L^X$ is the i^{th} node counted from X, where $i \ge 0$ and P_L^X is as defined in Property 1; then

$$n_i^L = N^{A}_{(k+i,2^{k}-1)} \in \begin{cases} T_{X_{2^{k-i}-1}}, 0 \le i \le k \\ T_{X_0}, i > k \end{cases}$$

Proof. Taking $j = 2^k - 1$ in Property 1 immediately yields

$$n_i^L = N^{A}_{(k+i,2^{k}-1)}$$

Note that

$$\begin{aligned} &\left\lfloor \frac{N^{A}_{(k+i,2^{k}-1)}-1}{2^{i}} \right\rfloor = \left\lfloor \frac{2^{k+i}(A-1)+2(2^{k}-1)}{2^{i}} \right\rfloor \\ &= \left\lfloor 2^{k}(A-1)+2(2^{k-i}-\frac{1}{2^{i}}) \right\rfloor = \left\lfloor 2^{k}(A-1)+2(2^{k-i}-1)+2-\frac{1}{2^{i-1}} \right\rfloor \\ &= \begin{cases} 2^{k}(A-1)+2(2^{k-i}-1), i=0 \\ 2^{k}(A-1)+2(2^{k-i}-1)+1, 1 \le i \le k \\ 2^{k}(A-1), i > k \end{cases} \end{aligned}$$

it is known by Lemma 1 that, on level k of T_A, the ancestor of $N^{A}_{(k+i,2^{k}-1)}$ is $N^{A}_{(k,2^{k-i}-1)}$ when $0 \le i \le k$ whereas it is $N^{A}_{(k,0)}$ when $i > k$.

Example 2. Again taking in T_{21} as an example. Take $k = 2$ and $j = 0$; then

$$\begin{aligned} n_0^R &= N^{21}_{(2+0,2^{2}-2^{2})} = N^{21}_{(2,0)} = 81 \in T_{X_0} \\ n_1^R &= N^{21}_{(2+1,2^{2+1}-2^{2})} = N^{21}_{(3,4)} = 169 \in T_{X_{2^2-2^{2-1}}} = T_{X_2} \\ n_2^R &= N^{21}_{(2+2,2^{2+2}-2^{2})} = N^{21}_{(4,12)} = 345 \in T_{X_{2^2-2^{2-2}}} = T_{X_3} \\ n_3^R &= N^{21}_{(2+3,2^{2+3}-2^{2})} = N^{21}_{(5,28)} = 697 \in T_{X_3} \end{aligned}$$

Take $k = 2$ and $j = 3$; then

$$n_0^L = N^{21}_{(2+0,2^2-1)} = N^{21}_{(2,3)} = 87 \in T_{X_{2^{2-0}-1}} = T_{X_3}$$
$$n_1^L = N^{21}_{(2+1,2^2-1)} = N^{21}_{(3,3)} = 167 \in T_{X_{2^{2-1}-1}} = T_{X_1}$$
$$n_2^L = N^{21}_{(2+2,2^2-1)} = N^{21}_{(4,3)} = 327 \in T_{X_0}$$
$$n_3^L = N^{21}_{(2+3,2^2-1)} = N^{21}_{(5,3)} = 645 \in T_{X_0}$$

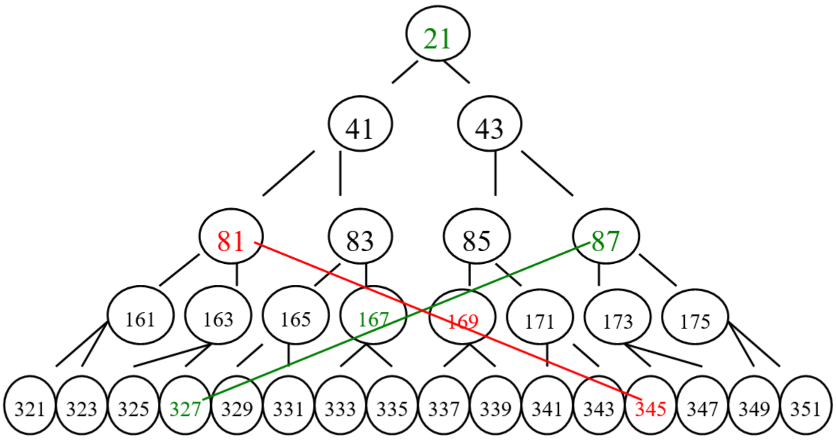

Fig. 5. Symmetric connections parallel to the borders of T_{21}

Remark 2. It can be seen that, by the identity $2^k - 1 - (\underline{2^k - 2^{k-i}}) = \underline{2^{k-i} - 1}$, $X_{2^{k-i}-1}$ and $X_{2^k - 2^{k-i}}$ are two symmetric nodes on level k of T_A.

Proposition 3. Let $X = N^A_{(k,j)}$ with $k > 0$ be a node of T_A, P_L^X and P_R^X be defined as those in Property 1; assume $n_i^L \in P_L^X$ and $n_i^R \in P_R^X$ are respectively the i^{th} nodes counted from X, where $i \geq 0$; then

$$n_i^L \in T_{X_{\omega_i}} \quad \text{and} \quad n_i^R \in T_{X_{2^k - 2^{k-i} + \omega_i}}$$

where $\omega_i = \left\lfloor \frac{j}{2^i} \right\rfloor$ and symbol X_α means $N^A_{(k,\alpha)}$.

Proof. Consider on level k of T_A the ancestors of n_i^L and n_i^R, respectively. By Property 1, it follows

$$\frac{n_i^L - 1}{2^i} = 2^k(A-1) + \frac{j}{2^{i-1}} \Rightarrow \left\lfloor \frac{n_i^L - 1}{2^i} \right\rfloor = 2^k(A-1) + \left\lfloor \frac{j}{2^{i-1}} \right\rfloor$$

and

$$n_i^R = 2^{k+i}(A-1) + 2(2^{k+i} - 2^k + j) + 1 \Rightarrow \left\lfloor \frac{n_i^R - 1}{2^i} \right\rfloor = 2^k(A-1) + 2(2^k - 2^{k-i}) + \left\lfloor \frac{j}{2^{i-1}} \right\rfloor.$$

Thus the quantity $\left\lfloor \frac{j}{2^{i-1}} \right\rfloor$ is the key to determine the ancestors. By Lemma 2, whether $\left\lfloor \frac{j}{2^{i-1}} \right\rfloor$ is even, say $\left\lfloor \frac{j}{2^{i-1}} \right\rfloor = 2\omega$, or it is odd, say $\left\lfloor \frac{j}{2^{i-1}} \right\rfloor = 2\omega + 1$, where $\omega \geq 0$ is an integer, it results in

$$A^i_{n_i^L} = 2^k(A-1) + 2\omega + 1$$

and

$$A^i_{n_i^R} = 2^k(A-1) + 2(2^k - 2^{k-i} + \omega) + 1$$

Referring to Lemma 2, it follows

$$A^i_{n_i^L} = 2^k(A-1) + 2\left\lfloor \frac{j}{2^i} \right\rfloor + 1$$

and

$$A^i_{n_i^R} = 2^k(A-1) + 2(2^k - 2^{k-i} + \left\lfloor \frac{j}{2^i} \right\rfloor) + 1$$

Since $A^i_{n_i^L}$ and $A^i_{n_i^R}$ are roots of subtrees from level k, the proposition is established.

Remark 3. It is seen that $\omega_i = 0$ when $i > \lfloor \log_2 j \rfloor$. There is always an i_0 such that $i > i_0$ leads to $n_i^L \in T_{X_0}$ and $n_i^R \in T_{X_{2^k-1}}$, where $X_0 = N^A_{(k,0)}$ and $X_{2^k-1} = N^A_{(k,2^k-1)}$.

Proposition 4. Let $X = N^A_{(k,j)}$ and $Y = N^A_{(k,2^k-1-j)}$ with $k > 0$ be two symmetric nodes on level k of T_A. Let P_L^X and P_R^X be the two type-1 connections starting from X and parallel respectively to the left and the right borders of T_A, P_L^Y and P_R^Y be the two type-1 connections starting from Y and parallel respectively to the left and the right borders of T_A. Assume $n_i^{LX} \in P_L^X$, $n_i^{RX} \in P_R^X$, $n_i^{LY} \in P_L^Y$ and $n_i^{RY} \in P_R^Y$ are nodes on the connections, where $i \ge 0$; then on level $k+i$ of T_A, n_i^{RX} is symmetric to n_i^{LY} and n_i^{LX} is symmetric to n_i^{RY}.

Proof. By Property 1, it follows

$$n_i^{RX} = N^A_{(k+i,2^{k+i}-2^k+j)} = 2^{k+i}(A-1) + 2(2^{k+i} - 2^k + j) + 1$$

and

$$n_i^{LY} = N^A_{(k+i,2^k-1-j)} = 2^{k+i}(A-1) + 2(2^k - 1 - j) + 1.$$

Let $\omega = 2^k - 1 - j$; then $2^{k+i} - 1 - \omega = 2^{k+i} - 2^k + j$. This immediately shows n_i^{RX} is symmetric to n_i^{LY}. Likewise, it can be proved that n_i^{LX} is symmetric to n_i^{RY}.

Property 2. Let $X = N^A_{(k,j)}$ and $Y = N^A_{(k,2^k-1-j)}$ with $k > 0$ and $X < Y$ be two symmetric nodes on level k of T_A; let P_{YR}^X be the type-2 connection starting from X and parallel to the right border of T_Y, P_{XL}^Y be the type-2 connection starting from Y and parallel to the left border of T_X. Assume $n_i^{YL} \in P_{XL}^Y$ and $n_i^{XR} \in P_{YR}^X$; then

$$n_i^{YL} = 2^i(X-1) + 2(2^k - 2j) - 1$$

and

$$n_i^{XR} = 2^i(Y+1) - 2(2^k - 2j) + 1$$

Proof. Let d be the distance from X to Y; then

$$d = \frac{Y-X}{2} + 1 = 2^k - 2j$$

For integer $i \geq 0$, the node on level i and on the left border of T_X is

$$N^X_{(i,0)} = 2^i(X-1)+1$$

The node on level i and on the right border of T_Y is

$$N^Y_{(i,2^i-1)} = 2^i(Y+1)-1$$

There by, n_i^{YL} and n_i^{XR} are necessary to satisfy

$$n_i^{YL} = N^X_{(i,0)} + 2(d-1) = 2^i(X-1)+1+2(2^k-2j-1)$$

and

$$n_i^{XR} = N^Y_{(i,2^i-1)} - 2(d-1) = 2^i(Y+1)-1-2(2^k-2j-1)$$

Proposition 5. Let $X = N^A_{(k,j)}$ and $Y = N^A_{(k,2^k-1-j)}$ with $k>0$ and $X<Y$ be two symmetric nodes on level k of T_A; let P^X_{YR} and P^Y_{XL} be the connections as defined in Property 2. Assume $n_i^{YL} \in P^Y_{XL}$ and $n_i^{XR} \in P^X_{YR}$; where $i \geq 0$; then

(1) n_i^{YL} and n_i^{XR} are symmetric nodes in T_A.
(2) $n_s^{YL} \in T_X$ and $n_s^{XR} \in T_Y$ when $s \geq k$.
(3) n_i^{YL} and n_i^{XR} are alternatively calculated by

$$n_i^{YL} = Y + (2^i-1)(X-1)$$

and

$$n_i^{XR} = X + (2^i-1)(Y+1)$$

Proof. By Property 2, it holds

$$n_i^{YL} = 2^i(X-1)+1+2(2^k-2j-1)$$

and

$$n_i^{XR} = 2^i(Y-1)+2(2^i-2^k+2j)+1$$

Since X and Y are symmetric, it follows $0 \leq j \leq 2^{k-1}-1$, $1 \leq 2^k-2j-1 \leq 2^k-1$ and $2^i-2^k \leq 2^i-2^k+2j \leq 2^i-2$. It is sure that $n_i^{YL} \in T_X$ and $n_i^{XR} \in T_Y$ when $i \geq k$, which validates the conclusion (2).

Note that

$$X = N^A_{(k,j)} = 2^k(A-1)+2j+1$$

and

$$Y = N^A_{(k,2^k-1-j)} = 2^k(A-1)+2(2^k-1-j)+1$$

Substituting these two into the expressions of n_i^{YL} and n_i^{XR} yields

$$n_i^{YL} = 2^{k+i}(A-1) + 2(2^k + 2^i j - 2j - 1) + 1$$

and

$$n_i^{XR} = 2^{k+i}(A-1) + 2(2^{k+i} - 2^k - 2^i j + 2j) + 1$$

Let $\omega = 2^k + 2^i j - 2j - 1$; then $2^{k+i} - 2^k - 2^i j + 2j = 2^{k+i} - 1 - (2^k + 2^i j - 2j - 1) = 2^{k+i} - 1 - \omega$ and it yields

$$n_i^{YL} = 2^{k+i}(A-1) + 2\omega + 1$$

and

$$n_i^{XR} = 2^{k+i}(A-1) + 2(2^{k+i} - 1 - \omega) + 1$$

Obviously, $n_i^{YL} \in T_A$ and $n_i^{XR} \in T_A$ if $0 \le \omega \le 2^{k+i} - 1$. In fact, direct calculation shows $\omega = 2^k - 1 - j$ and $2^{k+i} - 1 - \omega = j$ when $i = 0$. This means $n_0^{YL} \in T_A$ and $n_0^{XR} \in T_A$. When $i > 0$, it yields by $0 \le j \le 2^{k-1} - 1$

$$\begin{aligned} & 0 \le j \le 2^{k-1} - 1 \Rightarrow 0 \le 2^i j \le 2^{k+i-1} - 2^i \\ & \Rightarrow 1 \le \omega \le 2^{k+i-1} - 2^i + 2^k - 1 < 2^{k+i} - 1 \end{aligned}$$

and accordingly,

$$n_i^{YL} = N_{(k+i,\omega)}^A \in T_A$$

and

$$n_i^{XR} = N_{(k+i,2^{k+i}-1-\omega)}^A \in T_A$$

The conclusion (3) is simply proved by the following reasoning

$$\begin{aligned} & n_{i+1}^{YL} - n_i^{YL} = 2^i(X-1) \\ & n_1^{YL} - Y = 2^0(X-1) \\ & n_2^{YL} - n_1^{YL} = 2(X-1) \\ & n_3^{YL} - n_2^{YL} = 2^2(X-1) \\ & \ldots\ldots \\ & n_k^{YL} - n_{k-1}^{YL} = 2^{k-1}(X-1) \\ & \Rightarrow \\ & n_k^{YL} = Y + (2^k - 1)(X-1) \end{aligned}$$

$$\begin{aligned} & n_{i+1}^{XR} - n_i^{XR} = 2^i(Y+1) \\ & n_1^{XR} - X = 2^0(Y+1) \\ & n_2^{XR} - n_1^{XR} = 2(Y+1) \\ & n_3^{XR} - n_3^{XR} = 2^2(Y+1) \\ & \ldots \\ & n_k^{XR} - n_{k-1}^{XR} = 2^{k-1}(Y+1) \\ & \Rightarrow \\ & n_k^{XR} = X + (2^k - 1)(Y+1) \end{aligned}$$

Property 3. Let $X = N^A_{(k,j)}$ and $Y = N^A_{(l,s)}$ with $k > 0$ and $l - k = \delta > 0$ be two nodes of T_A ; assume $Y \notin T_X$ and Y is to the right of T_X. Let P^X_{YR} be the type-3 connection starting from X and parallel to the right border of T_Y, P^Y_{XL} be the type-3 connection starting from Y and parallel to the left border of T_X, as depicted in Fig. 3. Assume $n^{XL}_i \in P^Y_{XL}$ and $n^{YR}_i \in P^X_{YR}$; then for given an $i \geq 0$

$$n^{XL}_i = Y + 2^\delta(2^i - 1)(X - 1)$$

and

$$n^{YR}_i = (2^i - 1)(Y + 1) + 2^\delta(X + 1) - 1$$

Proof. Let d_l be the distance from Y to the left border of T_X, and d_r be the distance from Y to the right border of T_X ; then

$$d_l = \frac{Y - 2^\delta(X - 1) - 1}{2} + 1 \Rightarrow 2(d_l - 1) = Y - 2^\delta(X - 1) - 1$$

and

$$d_r = \frac{Y - 2^\delta(X + 1) + 1}{2} + 1 \Rightarrow 2(d_r - 1) = Y - 2^\delta(X + 1) + 1$$

It follows

$$n^{XL}_i = 2^{\delta+i}(X - 1) + 1 + 2(d_l - 1) = Y + 2^\delta(2^i - 1)(X - 1)$$

and

$$n^{YR}_i = 2^i(Y + 1) - 1 - 2(d_r - 1) = (2^i - 1)(Y + 1) + 2^\delta(X + 1) - 1$$

Example 3. Again taking in T_{21} as an example. Take $X = 41$ and $Y = 85$; then

$$n^{XL}_1 = 85 + 2^1(2^1 - 1)(41 - 1) = 165$$
$$n^{XL}_2 = 85 + 2^1(2^2 - 1)(41 - 1) = 325$$
$$n^{XL}_3 = 85 + 2^1(2^3 - 1)(41 - 1) = 645$$

$$n^{YR}_1 = (2^1 - 1)(85 + 1) + 2^1(41 + 1) - 1 = 169$$
$$n^{YR}_2 = (2^2 - 1)(85 + 1) + 2^1(41 + 1) - 1 = 341$$
$$n^{YR}_3 = (2^3 - 1)(85 + 1) + 2^1(41 + 1) - 1 = 685$$

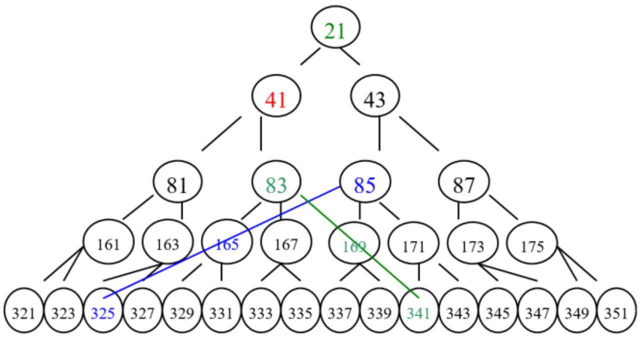

Fig. 6. Type-3 connections in T_{21}

Property 3*. Let $X = N^A_{(k,j)}$ and $Y = N^A_{(l,s)}$ with $k > 0$ and $l - k = \delta > 0$ be two nodes of T_A; assume $Y \notin T_X$ and Y is to the left of T_X. Let P^X_{YL} be the type-3 connection starting from X and parallel to the left border of T_Y, P^Y_{XR} be the type-3 connection starting from Y and parallel to the right border of T_X, as depicted in Fig. 7. Assume $n^{YL}_i \in P^X_{YL}$ and $n^{XR}_i \in P^Y_{XR}$; then for given an $i \geq 0$

$$n^{YL}_i = (2^i - 1)(Y - 1) + 2^\delta (X - 1) + 1$$

and

$$n^{XR}_i = Y + 2^\delta (2^i - 1)(X + 1)$$

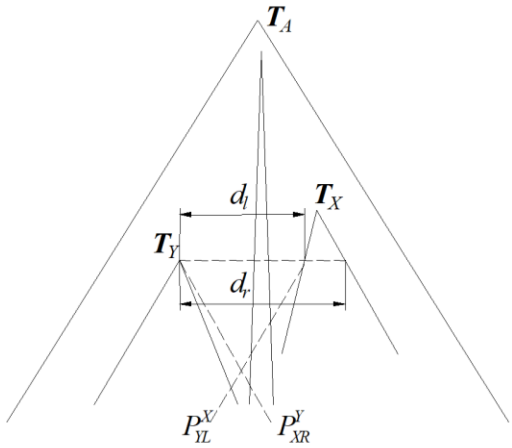

Fig. 7. Type-3 connections parallel to borders

Proof. Let d_l be the distance from Y to the left border of T_X and d_r be the distance from Y to the right border of T_X; then

$$d_l = \frac{2^\delta (X - 1) + 1 - Y}{2} + 1 \Rightarrow 2(d_l - 1) = 2^\delta (X - 1) + 1 - Y$$

and

$$d_r = \frac{2^\delta (X + 1) - 1 - Y}{2} + 1 \Rightarrow 2(d_r - 1) = 2^\delta (X + 1) - 1 - Y$$

It follows

$$n^{YL}_i = 2^i (Y - 1) + 1 + 2(d_l - 1) = (2^i - 1)(Y - 1) + 2^\delta (X - 1) + 1$$

and

$$n^{XR}_i = 2^{\delta + i}(X + 1) - 1 - 2(d_r - 1) = Y + 2^\delta (2^i - 1)(X + 1)$$

Proposition 6. Let $X = N^A_{(k,j)}$ and $Y = N^A_{(l,s)}$ with $k > 0$ and $l - k = \delta > 0$ be two nodes of T_A; assume $Y \notin T_X$ and Y is to the right of T_X. Let P^X_{YR} be the type-3 connection starting from X and parallel to the right border of T_Y, P^Y_{XL} be the type-3 connection starting from Y and parallel to the left border of T_X. Assume $n^{XL}_i \in P^Y_{XL}$ and $n^{YR}_i \in P^X_{YR}$; then $n^{XL}_i \in T_X$ with $i > \max(k, \delta)$ while $n^{YR}_s \in T_Y$ with $s \geq l$; and it holds

$$n_{i+1}^{XL} - n_i^{XL} = 2^{\delta+i}(X-1)$$

and

$$n_{i+1}^{YR} - n_i^{YR} = 2^i(Y+1) \Rightarrow n_j^{YR} = N_{(\delta,2^\delta-1)}^X + (2^j-1)(Y+1), j \geq 0$$

Proof. First, $l-k=\delta>0$ implies $0<\delta\leq l$ because it is contradictory that $l-k>l \Rightarrow k<0$. Now by Property 3, it holds

$$\frac{n_i^{XL}-1}{2^i} = \frac{Y-1-2^\delta(X-1)}{2^i} + 2^\delta(X-1)$$

and

$$\frac{n_i^{YR}-1}{2^i} = (Y+1) - \frac{Y+1-2^\delta(X+1)+2}{2^i}$$

Note that, referring to d_l and d_r defined in the proof of Property 3, it follows

$$2(d_l-1) = Y-2^\delta(X-1)-1$$

and

$$2(d_r-1) = Y+1-2^\delta(X+1)$$

As a result, it leads to

$$\frac{n_i^{XL}-1}{2^i} = \frac{d_l-1}{2^{i-1}} + 2^\delta(X-1)$$

and

$$\frac{n_i^{YR}-1}{2^i} = Y+1-\frac{d_r}{2^{i-1}}$$

Since

$$\begin{aligned}
&\frac{N_{(\delta,2^\delta-1)}^X - N_{(\delta,0)}^X}{2} + 1 \leq d_l \leq \frac{Y-N_{(\delta,0)}^X}{2} + 1 \\
&\Rightarrow 2^\delta + 1 \leq d_l \leq \frac{2^{l+2}-1-2^\delta(X-1)-1}{2} + 1 = 2^{l+1} - 2^{\delta-1}(X-1) \\
&\Rightarrow 2^\delta + 1 \leq d_l \leq 2^{l+1} - 2^{\delta-1}(2^{k+1}+1-1) = 2^{l+1} - 2^l = 2^l \\
&\Rightarrow 2^{\delta-i+1} \leq \frac{d_l-1}{2^{i-1}} \leq 2^{l-i+1} - \frac{1}{2^{i-1}}
\end{aligned}$$

and

$$1 \le d_r \le \frac{Y - N^X_{(\delta, 2^\delta - 1)}}{2} + 1$$
$$\Rightarrow 1 \le d_r \le \frac{2^{l+2} - 1 - 2^\delta (X+1) + 1}{2} + 1 = 2^{l+1} - 2^{\delta - 1}(X+1) + 1$$
$$\Rightarrow 1 \le d_r \le 2^{l+1} - 2^{\delta-1}((2^{k+1}+1)+1) + 1 = 2^{l+1} - 2^{k+\delta} - 2^\delta + 1 = 2^l - 2^\delta + 1$$
$$\Rightarrow \frac{1}{2^{i-1}} \le \frac{d_r}{2^{i-1}} \le 2^{l-i+1} - 2^{\delta - i + 1} + \frac{1}{2^{i-1}}$$
$$\Rightarrow -2^{l-i+1} + 2^{\delta-i+1} - \frac{1}{2^{i-1}} \le \frac{d_r}{2^{i-1}} \le -\frac{1}{2^{i-1}}$$

it yields

$$2^\delta (X-1) + \frac{2^\delta}{2^{i-1}} \le \frac{n_i^{XL} - 1}{2^i} \le 2^\delta (X-1) + \frac{2^l - 1}{2^{i-1}}$$

and

$$Y + 1 - \frac{2^l - 2^\delta}{2^{i-1}} \le \frac{n_i^{YR} - 1}{2^i} = Y + 1 - \frac{1}{2^{i-1}}$$

Obviously, on level δ of T_X , the ancestor of n_i^{XL} is to the right of $N^X_{(\delta,0)}$ and it lies in T_X if $i > \max(l-\delta, \delta) = \max(k, \delta)$. Similarly, the ancestor of n_i^{YR} is to the left of Y and it lies in T_Y if $i \ge l$.

Next the proof of the two equalities is just a simple reasoning like those in the proof of Proposition 1.

Proposition 6*. Let $X = N^A_{(k,j)}$ and $Y = N^A_{(l,s)}$ with $k > 0$ and $l - k = \delta > 0$ be two nodes of T_A ; assume $Y \notin T_X$ and Y is to the left of T_X. Let P^X_{YL} be the type-3 connection starting from X and parallel to the left border of T_Y, P^Y_{XR} be the type-3 connection starting from Y and parallel to the right border of T_X. Assume $n_i^{YL} \in P^X_{YL}$ and $n_i^{XR} \in P^Y_{XR}$; then $n_s^{YL} \in T_Y$ with $s \ge l$ while $n_i^{XR} \in T_X$ with $i > \max(k, \delta)$; and it holds

$$n_{i+1}^{XR} - n_i^{XR} = 2^{\delta+i}(X+1)$$

and

$$n_{i+1}^{YL} - n_i^{YL} = 2^i (Y-1) \Rightarrow n_j^{YL} = N^X_{(\delta,0)} + (2^j - 1)(Y-1)$$

Proof. (Omitted)

4 Applications in Integer Factorization

Connections make an outer node of a subtree be related with an inner node of the subtree; since it is easy for an inner node to trace up to reach its root, this thereby enables certain properties of the outer node to be associated with (transmitted to) the root of the subtree in the least searching steps. This section demonstrates such an operation.

4.1 General rule

The propositions proven previously reveal that, a node c on a connection starting from a node o can penetrate into an r-rooted subtree, as shown in Fig. 8, and the nodes o, c and r are related with the following equality

$$c = r + 2^\alpha (2^\beta - 1) o$$

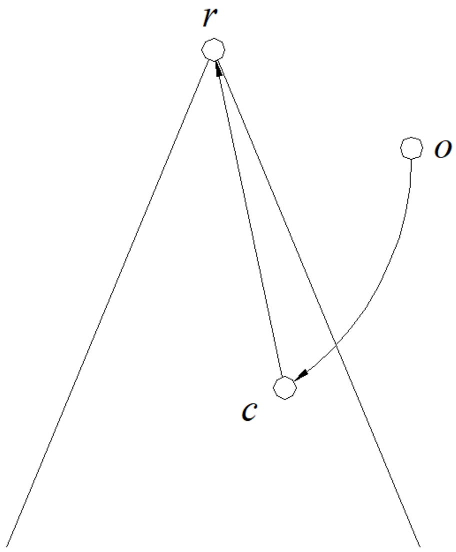

Fig. 8. Triangle relationship of connections

This relationship can be said to be a triangle relationship of connections and it can derive many amazing results by means of evaluating different values to *r* and *o*. This idea is the general rule of connecting different nodes in a valuated binary tree.

4.2 Factorization of odd integers

With the triangle relationship of the connections, this section shows a divisor of the form $2^{\alpha}+1, 2^{\alpha}-1$, $\gamma+2^{\alpha}(2^{\beta}-1)$ or $\gamma+2^{\alpha}(2^{\beta}-1)\lambda$ can be easily found out in a positive composite odd integer, as the following corollaries state.

Corollary 1. Let $m = pq > 3$ be an odd composite positive integer and the divisor *p* is of the form $2^{\alpha}-1$ with integer $\alpha > 0$; then *p* can be found out in $O(\lfloor \log_2 p \rfloor + 2)$ searching steps.

Proof. By Lemma 1, *m* lies on level $k = \lfloor \log_2 m \rfloor - 1$ of T_3. If *m* lies on the left border of T_3, construct a sequence M_R by

$$M_R = \{m + 2^{k+2}(2^1-1), m + 2^{k+2}(2^2-1), ..., m + 2^{k+2}(2^i-1), ...\}$$

If it lies on the right border of T_3, construct M_L by

$$M_R = \{m + 2^{k+1}(2^1-1), m + 2^{k+1}(2^2-1), ..., m + 2^{k+1}(2^i-1), ...\}$$

If it is an intermediate node, construct

$$M_l = \{m + 2^{k+1}(2^1-1), m + 2^{k+1}(2^2-1), ..., m + 2^{k+1}(2^i-1), ...\}$$

or

$$M_r = \{m + 2^{k+2}(2^1-1), m + 2^{k+2}(2^2-1), ..., m + 2^{k+2}(2^i-1), ...\}$$

It can be seen by Proposition 1 that, for each case there must be an α such that $p = \gcd(m_\alpha, m)$, where $m_\alpha = m + 2^{k+2}(2^\alpha - 1)$ or $m_\alpha = m + 2^{k+1}(2^\alpha - 1)$. Since $p = 2^\alpha - 1$, it follows $\alpha = \lfloor \log_2 p \rfloor + 2$.

Corollary 2. Let $m = pq > 3$ be an odd composite positive integer and the divisor p is of the form $2^\alpha + 1$ with integer $\alpha > 0$; then p can be found out in at most $O(\lfloor \log_2 p \rfloor + 1)$ searching steps.

Proof. By Lemma 1, m lies on level $\lfloor \log_2 m \rfloor - 1$ of T_3. Consider the case that it is an intermediate node. Refer to the proof of Corollary 1 and construct

$$M_l = \{m + 2(2^2 - 1), ..., m + 2(2^{2i} - 1), ...\}$$

it is seen that must be an α such that $p = \gcd(m_\alpha, m)$, where $m_\alpha = m + 2^2(2^{2\alpha} - 1)$ or $m_\alpha = m + 2(2^{2\alpha} - 1)$.

Remark 4. Corollaries 1 and 2 look very trivial because they seem so elementary that any one might be able to think of the constructions. However, they are here derived *theoretically* from Proposition 1. In other words, they are rather theoretical results than something someone thinks of.

Corollary 3. Suppose the divisor p of the positive composite odd $n = pq$ is of the form $\gamma + 2^\alpha(2^\beta - 1)$, where $\alpha > 0$, $\beta > 0$ are integers, and $q > 1$, $\gamma \ge 1$ are odd integers; then q can found out in at least $O(\log_2 p)$ and at most $O((\frac{\log_2 p + 1}{2})^2)$ searching steps.

Proof. Take the triangle formula $n_i^{XL} = Y + 2^\delta(2^i - 1)(X - 1)$ established in Property 3 as an example to have an analysis. Let $X - 1 = 2^\sigma s$, where $\sigma > 0$ and s is a positive odd integer. For convenience denote n_i^{XL} by n; then $n = Y + 2^{\delta+\sigma}(2^i - 1)s$

Take an arbitrary positive odd integer Y satisfying $Y = \gamma s$ and $Y > 2X + 1$; denote $\gamma + 2^{\delta+\sigma+i} - 2^{\delta+\sigma} = u$; then

$$n = (\gamma + 2^{\delta+\sigma+i} - 2^{\delta+\sigma})s = us$$

By Property 3, there is an i_0 such that $n = n_i^{XL} \in T_X$ when $i \ge i_0 > 0$. Consequently, n is a descendant of X. Thus X can be found by searching in the direct ancestors of n, and then s can be found by $s = \frac{X-1}{2^\sigma}$. Note that, there are $\lambda = \lfloor \log_2 n \rfloor - \lfloor \log_2 X \rfloor$ levels from n to X, and it takes σ steps to calculate s with X. It is sure that the total searching steps are

$$t = \lambda\sigma = (\lfloor \log_2 n \rfloor - \lfloor \log_2 X \rfloor)\sigma$$

Since

$$\lfloor \log_2 X \rfloor = \lfloor \log_2(2^\sigma s + 1) \rfloor = \sigma + \left\lfloor \log_2\left(s + \frac{1}{2^\sigma}\right) \right\rfloor = \sigma + \lfloor \log_2 s \rfloor$$

it follows

$$t = \sigma(\lfloor \log_2 n \rfloor - \lfloor \log_2 s \rfloor - \sigma) \le \sigma(\log_2 n - \log_2 s + 1 - \sigma) = \sigma(\log_2 u + 1 - \sigma)$$

Next is to show $\sigma \le \log_2 u$. This can be done with the proof by contradiction. Assume $\sigma > \log_2 u$; then it follows $\sigma > \log_2 u \Rightarrow u < 2^\sigma \Rightarrow X = 2^\sigma s + 1 > us + 1 \Rightarrow X > n + 1$

which is a contradiction. Consequently, under the condition $\sigma \le \log_2 u$, it holds

$$\log_2 u \le t \le (\frac{\log_2 u + 1}{2})^2$$

The corollary surely holds by substituting *s* with *q* and *u* with *p* in the above reasoning process.

Corollary 4. Suppose the divisor *p* of the positive composite odd $n = pq$ is of the form $\gamma + 2^{\alpha}(2^{\beta} - 1)\lambda$, where $\alpha > 0, \beta > 0$ are integers, and $q > 1$, $\gamma \ge 1$, $\lambda \ge 1$ are odd integers; then *q* can found out in at most $O(\log_2 p)$ searching steps.

Proof. Take the triangle formula $n_i^{XL} = Y + 2^{\delta}(2^i - 1)(X - 1)$ established in Property 3 as an example. Assume $X - 1 = 2^{\sigma}\lambda s$ and $Y = \gamma s$, where integer $\sigma > 0$ and $\lambda \ge 1, \gamma \ge 1$ are odd integers; denote n_i^{XL} by *n*. Then

$$n = (\gamma + 2^{\delta+\sigma}(2^i - 1)\lambda)s = us$$

where $u = \gamma + 2^{\delta+\sigma}(2^i - 1)\lambda$.

Since $n \in T_X$ for some i , tracing up from n surely reaches *X* and thus *s* is the common divisor of $X - 1$ and *n*. The searching step from *n* to *X* is $t = \lfloor \log_2 n \rfloor - \lfloor \log_2 X \rfloor$. It can be proven $t \le \lfloor \log_2 u \rfloor$. In fact, assumption of $t > \lfloor \log_2 u \rfloor$ yields

$$\begin{aligned}
& t = \lfloor \log_2 n \rfloor - \lfloor \log_2 X \rfloor > \lfloor \log_2 u \rfloor \\
& \Rightarrow \lfloor \log_2 n \rfloor - \lfloor \log_2 u \rfloor > \lfloor \log_2 X \rfloor \\
& \Rightarrow 1 + \lfloor \log_2 n - \log_2 u \rfloor > \lfloor \log_2 X \rfloor \\
& \Rightarrow \lfloor \log_2 s \rfloor - \lfloor \log_2 X \rfloor + 1 > 0
\end{aligned}$$

It is contradictory because $X = 2^{\sigma}\lambda s + 1 \Rightarrow \lfloor \log_2 X \rfloor = \lfloor \log_2(2^{\sigma}\lambda s + 1) \rfloor \ge \sigma + \lfloor \log_2 \lambda \rfloor + \lfloor \log_2 s \rfloor$.

Take the triangle formula $n_i^{XL} = Y + 2^{\delta}(2^i - 1)(X + 1)$ in Property 3* can also derive this corollary.

4.3 Numerical experiments

Experiments for testing Corollary 4 are made to factorize odd integers that are of the length from 101 to 105 decimal digits. Table 1 lists the experimental results. In the table, the column ‘Big Number *N*’ is the big odd composite number to be factorized, the column ‘nDigits’ is the number of decimal digits, the column ‘Found Divisor’ is the found divisor of N, the column ‘Tsteps’ is the number of searching steps calculated theoretically from the previous corollaries and the column ‘Rsteps’ is the real searching steps recorded by the computer. It can be seen that the real searching steps are within the bounds of the orifical searching steps in each case. For readers to know the algorithms more deeply, the Maple programs are list in the appendix section. Readers can test them with the programs.

Table 1. Experimental Results for Testing Corollary 4

Big Number *N*	**nDigits**	**Found Divisor**	**Tsteps**	**Rsteps**
38438269293846104157664192464697271137497178448383674628658272942798389824657295879238987183468752803	101	737779468553701220405796500475514437 9743	201	138
472373911803885589134770263222665653350941303069050338104094 5	102	110666920283055183060869475071327165700 77	204	138

Big Number ***N***	**nDigits**	**Found Divisor**	**Tsteps**	**Rsteps**
163836521093280851652408996506181118 49897				
209030087090003249475422808302558123 72227501841380260350883419 796076392339838633561706607281441501 47127	103	258222813993795427142028775 16643005397757	205	140
192061675199323399490464762999390147 226623529505541524829224599 583516268996902466878136494395647449 78023	104	472178859874368781059709760 304329243474189	204	144
454498542002546219985842043143542132 16503497614531938387270234 041366691345735139671704580663847398 7045863	105	377743087899495024847767808 2434633948186509	206	147

5 Assessments, Conclusions and Future Work

5.1 Assessments of the new results

Referring to paper [1], it is seen that three corollaries were proven and those corollaries were quite like the corollaries proven in 4.2. Nevertheless, there are differences. Comparing Corollary 1 of this paper to that Corollary 1, it can be seen that, the two corollaries state two different results in finding a divisor of a positive odd integer that has a divisor of the form $2^\alpha - 1$. For an odd composite positive integer $m = pq > 3$ whose divisor p is of the form $2^\alpha - 1$, this Corollary 1 shows p can be found out in $O(\lfloor \log_2 p \rfloor + 2)$ searching steps, whereas that Corollary 1 showed q could be found out within $O(1 + \log_2 p)$ under the condition $\alpha \ge \left\lfloor \frac{\log_2 m}{2} \right\rfloor + 1$. Obviously, this Corollary 1 is more flexible and applicable. It can also be seen that, this Corollary 3 is more flexible than that Corollary 3 although the two corollaries state the same topic in finding a divisor of a positive odd integer that has a divisor of the form $\gamma + 2^\alpha(2^\beta - 1)$. Finally, in paper [1] there was no Corollary 4, which is actually more extensive than Corollary 3. Consequently, it can be concluded that, the investigation in this paper is more subtle and beneficial for the researching purpose.

5.2 Conclusions and expectations

The valuated binary tree method demonstrated more and more attractive and reliable results in analyzing odd integers. Especially the application of geometric means enables it easy and clear to set up kinds of relationships among subtrees and nodes. This derives a very simple and fast way tp factorize the kind of odd composite integers. However, one thing should be told the readers here. That is that failures occur in factoring an $n = pq$ when both p and q are of the form $\gamma + 2^\alpha(2^\beta - 1)\lambda$. This remains further studies. Hope more young to join the work.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Wang X, LUO J, Tian Y, Ma L. Connections on valuated binary tree and their applications in factoring odd integers. Asian Research Journal of Mathematics. 2021;17(3):134-153.
DOI: 10.9734/arjom/2021/v17i330287

[2] Wang X. Valuated binary tree: A new approach in study of integers. International Journal of Scientific and Innovative Mathematical Research. 2016;4(3):63-67.
DOI: http://dx.doi.org/10.20431/2347-3142.0403008

[3] Wang X. Genetic traits of odd numbers with applications in factorization of integers. Global Journal of Pure and Applied Mathematics. 2017;13(2):493-517.
Available:http://www. ripublication.com/gjpam17/gjpamv13n2_29.pdf
Wang X. T3 tree and its traits in understanding integers. Advances in Pure Mathematics. 2018;8(5):494-507.
Available:https://doi.org/10.4236/apm.2018.85028

[4] Wang X, Guo H. Some divisibility traits on valuated binary trees. International Journal of Applied Physics and Mathematics. 2019;9(1):1-15.
Available:https://doi:org/10.17706/ijapm.2019.9.4.173-181

[5] Xingbo Wang. Fast approach to factorize odd integers with special divisors. Journal of Mathematics and Statistics. 2020;16(1):24-34.
Available:https://doi.org/10.3844/jmssp.2020.24. 34

[6] Xingbo Wang. Algorithm available for factoring big Fermat numbers. Journal of Software. 2020;15(3): 86-97.
Available:https://doi.org/ 10.17706/jsw.15.3.86-97

[7] Wang X. Bound estimation for divisors of RSA modulus with small divisor-ratio. International Journal of Network Security. 2021;23(3):412-425.

[8] Wang X, Zheng C. Some miscellaneous properties of valuated binary tree. Journal of Mathematics Research. 2021;13(3):1-12.
Available:https://doi.org/10.5539/jmr.v13n3p1

Appendix

Maple Source Codes

SubRoutine Father (Calculate the father of a node Son)

```
Father: =proc( Son)
local X, r;
r: =modp(Son,4);
   if r=1 then X: = (Son+1)/2;
   else    X: = (Son -1)/2;
   fi
End proc
```

MainRoutine Findq (Calculate the divisor q of odd composite integer *N*)

```
Findq: =proc(N)
   local X,T, AA, g, p,q, Tsteps, Rsteps:=0, len;
 AA: =Father(N);
 g: =gcd (AA, N);
 while g=1 do
  Rsteps: =Rsteps+1;
  T:=AA;
  X:=T-1;
  g: =gcd (X, N);
   AA: =Father(T);
 od;
  q: =g; p: =N/q;
 Tsteps: =floor(evalf(log2(p))
lprint ("Find q=", q, "Tsteps=", Tsteps, "Rsteps=", Rsteps);
End proc
```


Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle4.com/review-history/70120