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On Absolute Continuity of Non Negative Functions

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

This paper is committed to the study of absolute continuity of non negative functions with respect to vector measures. Almost everywhere properties are applied to establish boundedness, measurability and convergence of sequences of measurable functions. The measurability of sets with respect to vector duality functions with values in a Hilbert space is considered.

Keywords: Measurable sets; absolute continuity; integrable functions, non-negative functions functions.

1 Introduction

Many studies have been done on absolute continuity with respect to locally convex topological vector spaces under the conditions of finiteness and variation of vector measures. Other scholars applied measures of bounded variation with values in Normed linear spaces. In this paper, we consider absolute continuity of non negative functions. Properties of vector duality set functions with values in the product Banach spaces of absolutely summable functions $(\beta_{\epsilon_i} : i \in I)$ in X defined on the indexed set I are applied. Throuhout this paper, $(X \times Y, Z)$ denotes a bilinear system where $X \times Y$ is the product of Banach spaces X and Y and Z is a Hilbert space, (S, ρ) and (T, ϵ) denote measurable spaces with ρ and ϵ being the sigma rings of subsets of S

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and T respectively and $\mu : \rho \to \beta_{\epsilon_i}$ and $\nu : \epsilon \to Y$ denote vector measures where $\mu(E) = \sum_{i \in I} \mid \epsilon_i \mid (E) \in \beta_{\epsilon_i}$ and $\nu(F) \in Y$ for sets $E \in \rho$ and $F \in \epsilon$, $L'(\mu)$ and $L'(\nu)$ denotes first integral with respect to μ and ν repectively.

If ψ is a Z-valued bilinear function defined on $X \times Y$ such that $\psi : X \times Y \to Z$, then

$$< (\mu(E) \times \nu(F))_\psi, z^* > = < (\textstyle\sum_{i \in I} \mid \epsilon_i \mid (E) \times \nu(F))_\psi, z^* >$$

for each $i \in I$ where z^* is an element in Z^* the dual space of the Hilbert space Z is called vector duality function.

2 Basic Concepts

Definition 1 (Absolute continuity):

Let $\mu : \rho \to \beta_{\epsilon_i}$ and $\nu : \epsilon \to Y$ be vector measures. If α and β are non-negative set functions defined on ρ and ϵ respecively, then $\alpha \times \beta$ is absolutely continuous with respect to $\mu \times \nu$ if for each $\lambda > 0$ there corresponds a $\delta > 0$ such that $\mu \times \nu(E \times F) < \delta$ implies that $\alpha \times \beta(E \times F) < \lambda$ for every $E \times F \in \rho \times \epsilon$. We therefore write $\alpha \times \beta < \mu \times \nu$

Definition 2 (Almost uniformly convergence)

A sequence (f_n) of $X \times Y$ valued functions is said to $(\mu \times \nu)$ - converge to f almost uniformly if given $\lambda > 0$, there exists

$E \times F = (E \times F)(\epsilon) \in \rho \times \epsilon$ such that $\mu \times \nu(E \times F) < \lambda$ and

$\mid f_n(s,t) - f(s,t) \mid \to 0$ uniformly on $S \times T \setminus E \times F$

Definition 3 (Measurable function)

A function $f : S \times T \to X \times Y$ is said to be $(\mu \times \nu, X \times Y)$ - measurable

if and only if

i) Range$(f) \subset X \times Y$

ii) There exists a sequence (f_n) of $X \times Y$ valued functions converging $(\mu \times \nu, X \times Y)$ - a.e. to f

3 Results

The following propositions provide insights into properties of absolute continuity of non-negative functions.

Proposition 1

Let (S, ρ) and (T, ϵ) be measurable spaces, $(X \times Y, Z)$ a bilinear system and $\mu : \rho \to \beta_{\epsilon_i}$ and $\nu : \epsilon \to Y$ be vector measures.If α and β are non-negative measures defined on ρ and ϵ respectively such that
$\alpha \times \beta << \mu \times \nu$, then $\alpha << \mu$ and $\beta << \nu$

Proof: Let $G = E \times F \in \rho \times \epsilon$, $\lambda > 0$ and $\delta > 0$ such that $\mu(E) < \delta$ imply that $\alpha(E) < \lambda$ for any set $E \in \rho$ and $\nu(F) < \delta$ imply that $\beta(F) < \lambda$ for any set $F \in \epsilon$. Since $\alpha \times \beta << \mu \times \nu$, on application of properties of product measures [1] , we obtain

$$< (\mu \times \nu)(G), z^* > = < (\mu \times \nu)(E \times F), z^* >< \delta^2 \text{ implies that}$$

$$< (\alpha \times \beta)(G), z^* > = < (\alpha \times \beta)(E \times F), z^* >< \lambda^2$$

Consider the function $f : S \times T \to X \times Y$.For a fixed $s \in S$, then $f(s) \in L'(\nu)$. Let $\bigtriangledown^t = (s \in S : \nu_{f(s)}[G^s] < \delta)$ be the t - section of the set $\bigtriangledown$ [2] . It follows that,

$$\delta^2 \geq < (\mu \times \nu_{f(s)})(G), z^* > = < \mu(G^t) \times \nu_{f(s)}(G^s), z^* >.$$

Since $\nu_{f(s)}(G^s) > \delta$ on the complement of $\bigtriangledown^t$ in G^t, it follows that

$$\delta^2 \geq < (\mu(G^t) \times \nu_{f(s)}(G^s)), z^* >> \delta\mu((\bigtriangledown^t))^c$$

where $(\bigtriangledown^t))^c$ denotes the complement of $\bigtriangledown^t$

Therefore, $\mu((\bigtriangledown^t))^c < \delta$ implies $\alpha((\bigtriangledown^t))^c < \lambda$ i.e. $\alpha << \mu$

Similarly for a fixed $t \in T$, we have

$f(t) \in L'(\alpha)$. Let $\bigtriangledown^s = (t \in T : \alpha_{f(t)}[G^t] < \lambda)$. Therefore,

$$\lambda^2 \geq < (\alpha_{f(t)} \times \beta)(G), z^* > = < \alpha_{f(t)}(G^t) \times \beta(G^s), z^* > .$$

Hence, $\lambda^2 \geq < \alpha_{f(t)}(G^t) \times \beta(G^s), z^* >> \lambda\beta((\bigtriangledown^s))^c$

where $(\bigtriangledown^s))^c$ denotes the complement of $\bigtriangledown^s$

Therefore, $\beta((\bigtriangledown^s))^c < \lambda$ when $\nu((\bigtriangledown^s))^c < \delta$ i.e. $\beta << \nu$

Proposition 2: Let $(X \times Y, Z)$ a bilinear system, where X and Y are Banach spaces and Z is a Hilbert space. Let $\beta : \epsilon \to Y$ be a vector measure such that such that $\alpha \times \beta$ exists for every $\alpha : \rho \to \beta_{\epsilon_i}$. If $\alpha \times \beta << \mu \times \nu$ and $(\beta(F))_{\epsilon_i(E)} = LUB_n \sum_{i \in I} \sum_{k=1}^{n} \mid \epsilon_i \mid (E)\beta(F_k)$ where (F_k) is the partition of F, then $(\beta(F))_{\epsilon_i(E)} << \mu \times \nu$

Proof: Let α be a measure defined on a set $(\beta_{\epsilon_i} : i \in I)$ of absolutely summable functions $(\epsilon_i : i \in I)$ in X defined on an indexed set I. Since $\alpha \times \beta$ is absolutely continuous with respect to $\mu \times \nu$, given $\lambda > 0$ there exists

$$\delta = \delta(\epsilon) > 0 \text{ such that } < \mu \times \mu(G), z^* >< \delta \text{ implies } < \alpha \times \beta(G), z^* >< \lambda$$

for every $G \in \rho \times \epsilon$. Let $E \in \rho$ and $F \in \epsilon$ such that for $k > 0$ we have

$$\mu(E) < k \text{ and } \mu(F) < \delta k^{-1}$$

Let $(B(F))_{\epsilon_i(E)} = LUB_n \sum_{i \in I} \sum_{k=1}^{n} \mid \epsilon_i \mid (E)\beta(F_k) \in Z$ where (F_k) is the partition of F for $1 \leq k \leq n$ (Otanga *et al.*, 2015a). Define

$\alpha(E) = \sum_{i \in I} \mid \epsilon_i \mid (E)$ for any measurable set E [3]. If $G = E \times F \in \rho \times \epsilon$, then $G = \bigcup_{k=1}^{n} E \times F_k$. Therefore

$$< (\mu \times \nu)(G), z^* > = \sum_{k=1}^{n} < \mu(E)\nu(F_k), z^* > \leq \sum_{k=1}^{n} k < \nu(F_k), z^* >$$

$$= k < \nu(\bigcup_{k=1}^{n} F_k), z^* > = k < \nu(F), z^* > < \delta$$

Since $< (\alpha \times \beta)(G), z^* > < \lambda$, it follows that

$$< (\alpha \times \beta)(G), z^* > = \sum_{k=1}^{n} < \alpha(E)\beta(F_k), z^* >$$

$$= < \sum_{i \in I} \sum_{k=1}^{n} \mid \epsilon_i \mid (E)\beta(F_k), z^* >$$

Taking the least upper bound of right hand side of the equation

[4] , we obtain

$< (B(F))_{\epsilon_i(E)}, z^* > < \lambda$. Hence $(B(F))_{\epsilon_i(E)} << \mu \times \nu$

Proposition 3: Let $(X^{\epsilon_i} \times Y, z)$ be a bilinear system and α be a measure defined on a set $(\beta_{\epsilon_i} : i \in I)$ of absolutely summable functions $(\epsilon_i : i \in I)$ in X defined on an indexed set I. Let $\mu : \rho \to \beta_{\epsilon_i}$ and $\nu : \epsilon \to Y$ be vector measures. If for each $i \in I$, α_i and β_i are non-negative set functions defined on ρ and ϵ respectively such that $\alpha_i << \mu$ and $\beta_i << \nu$, then $\sum_{i \in I} \alpha_i \times \beta_i << \mu \times \nu$.

Proof: For each $E \in \rho$ and $F \in \epsilon$, let $\mu(E) = \sum_{i \in I} \mid \epsilon_i \mid (E) \in X^{\epsilon_i}$ and $\nu(F) \in Y$ such that $\mu(E)\nu(F) = \sum_{i \in I} \mid \epsilon_i \mid (E)\nu(F)$. For each $i \in I$, let $\alpha_i \times \beta_i << \mid \epsilon_i \mid \times \nu$ where $\alpha \times \beta_i$ is a non-negative set function on $\rho \times \epsilon$.

For each measurale set $E \times F$ and each $\lambda > 0$ there exists $\delta > 0$ such that [5]

$<\mid \epsilon_i \mid \times \nu(E \times F), z^* > < \delta$ implies $< (\alpha_i \times \beta_i)(E \times F), z^* > < \lambda$. Let

$\sigma \subset I$ be an arbitrary finite subset such that

$\sum_{i \in \sigma} < (\alpha_i \times \beta_i)(E \times F), z^* > = \sum_{i \in I} <\mid \epsilon_i \mid \times \nu(E \times F), z^* >$.

If $\sum_{i \in I} \alpha_i \times \beta_i$ is a set function defined on $\rho \times \epsilon$ by the formula

$\sum_{i \in I} < (\alpha_i \times \beta_i)(E \times F), z^* > = sup(\sum_{i \in \sigma} < (\alpha_i \times \beta_i)(E \times F), z^* >$

then for each $\lambda > 0$ there exists $\delta > 0$ such that

$\sum_{i \in I} <\mid \epsilon_i \mid \times \nu(E \times F), z^* > < \delta$ implies that

$\sum_{i \in I} < (\alpha_i \times \beta_i)(E \times F), z^* > < \lambda$

Hence $\sum_{i \in I} \alpha_i \times \beta_i << \mu \times \nu$

Proposition 4: Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions such that $f_n : S \times T \to X \times Y$ for each n. Let $\alpha : \rho \to X$ and $\beta : \epsilon \to Y$ be a vector measures such that such $\alpha \times \beta << \mu \times \nu$ where ν is a non-negative set function defined on ϵ.

If $f_n \to f$ $(\mu \times \nu, X \times Y)$-almost uniformly, then $f_n \to f$ almost everywhere. If f_n is $(\mu \times \nu, X \times Y)$-integrable, then f is integrable and

$< \int \mu_{|f_n(t)-f(t)|}((G')^t)\delta\nu(t), z^* >< \lambda$ for all $n \geq \aleph$, $\lambda > 0$, $t \in T$ and

$(G')^t \in \rho$.

Proof: Since $f_n \to f$ $(\mu \times \nu, X \times Y)$-almost uniformly, let G_m be a measurable set with respect to $\rho \times \epsilon$ such that $(\alpha \times \beta)(G_m) < \lambda \backslash 2m$ for each positive integer m and $\lambda > 0$. Let $f_n(s,t) \to f(s,t)$ uniformly on $S \times T \setminus G_m$. It follows that $G = \bigcap_{m=1}^{\infty} G_m$ is a $\alpha \times \beta$-null set and $f_n(s,t) \to f(s,t)$ foreach $(s,t) \in S \times T \setminus G_m$. Therefore $f_n \to f$ a.e. Since f is a limit of an f_n is $(\mu \times \nu, X \times Y)$-integrable function, then it is $(\mu \times \nu, X \times Y)$-integrable.

Since $\alpha \times \beta << \mu \times \nu$ (by hypothesis), given $\lambda > 0$ there corresponds a $\delta > 0$ such that $(\mu \times \nu)(G') < \delta$ implies $(\alpha \times \beta)(G') < \lambda \setminus 2m$ for every $G' \in \rho \times \epsilon$ amd $m > 0$. Since $f_n \to f$ $(\mu \times \nu, X \times Y)$-almost uniformly, there exists $G'' \subset G'$ such that for a fixed $t \in T$, we have

$$\alpha((G'')^t) = \sum_{i \in I} \mid \epsilon_i \mid ((G'')^t) < \lambda \setminus 2m$$

For all $n > \aleph$ and as a consequence of integral representation of product vector measure duality [6] , we have

$$< \int \mid f_n(t) - f(t) \mid \delta\mu, z^* >< \lambda \setminus 2 \sum_{i \in I} \mid \epsilon_i \mid ((G')^t \setminus (G'')^t)$$

It follows from measurable concepts in [7] that

$$< \int \mu_{|f_n(t)-f(t)|}((G')^t)\delta\nu(t), z^* >$$

$$\leq \lambda(\sum_{i \in I} \mid \epsilon_i \mid ((G')^t \setminus (G'')^t) \setminus 2 \sum_{i \in I} \mid \epsilon_i \mid ((G')^t \setminus (G'')^t) < \lambda \setminus 2$$

Since f_n is $(\mu \times \nu, X \times Y)$-integrable function, it is bounded.

Suppose $\int f_n(t)\delta\mu \leq m \setminus 2$ for any positive integer $m > 0$ and for a fixed $t \in T$ [8]. Then $f_n \to f$ implies that $\int \mid f_n(t) - f(t) \mid \delta\mu \leq m$ for all n.

Let Δ be a measurable set with respect to $\rho \times \epsilon$ such that $G' \setminus \Delta$ is a $\alpha \times \beta$-null set. On application of integral properties of vector measure

[9] and Yaogan, 2013), we obtain

$$< \int \mu_{|f_n(t)-f(t)|}((G'')^t)\delta\nu(t), z^* > = < \int \mu_{|f_n(t)-f(t)|}((G'')^t \bigcap \Delta^t)\delta\nu(t), z^* >$$

$$+ < \int \mu_{|f_n(t)-f(t)|}((G'')^t \setminus \Delta^t)\delta\nu(t), z^* >$$

Since $(G'')^t \setminus \Delta^t$ is a α-null set, therefore

$$< \int \mu_{|f_n(t)-f(t)|}((G'')^t)\delta\nu(t), z^* > \leq < \int \mu_{|f_n(t)-f(t)|}((G'')^t \bigcap \Delta^t)\delta\nu(t), z^* >$$

$$\leq m \sum_{i \in I} \mid \epsilon_i \mid (G'')^t \bigcap \Delta^t)$$

Since $\sum_{i \in I} \mid \epsilon_i \mid (G'')^t \bigcap \Delta^t) \leq \sum_{i \in I} \mid \epsilon_i \mid (G'')^t)$, it follows that

$< \int \mu_{|f_n(t)-f(t)|}((G'')^t)\delta\nu(t), z^* > \leq m \sum_{i \in I} | \epsilon_i | (G'')^t)$

$< m * \lambda \setminus 2m = \lambda \setminus 2$

Consequently,

$< \int \mu_{|f_n(t)-f(t)|}((G')^t)\delta\nu(t), z^* > \leq < \int \mu_{|f_n(t)-f(t)|}((G')^t \setminus (G'')^t)\delta\nu(t), z^* >$

$+ < \int \mu_{|f_n(t)-f(t)|}((G'')^t \setminus \Delta^t)\delta\nu(t), z^* >$

$< \lambda \setminus 2 + \lambda \setminus 2 = \lambda$

Corollary: Let $(X \times Y, Z)$ a bilinear system, where X and Y are Banach spaces and Z is a Hilbert space. Let $(S \times T, \rho \times \epsilon)$ be a measurable space and $(f_{G_n})_{n=1}^{\infty}$ be a sequence of $(\mu \times \nu, \beta_{\epsilon_i} \times Y)$ - a.e. bounded functions such that $f_{G_n} : S \times T \rightarrow X \times Y$ for each n. If $f_{G_n} \rightarrow f_G$ is $(\mu \times \nu, \beta_{\epsilon_i} \times Y)$ - almost uniformly where $G_n \uparrow G$ and $G, G_n \in \rho \times \epsilon$, then $LUB_n \sum_{i \in I} < (| \epsilon_i | \times \beta)(G_n), z^* > = \sum_{i \in I} < (| \epsilon_i | \times \beta)(G), z^* >$

Proof: $f : S \times T \rightarrow X \times Y$ and $f_G = \chi_G$ where $G \in \rho \times \epsilon$. Let $f_G \subseteq \beta_i$

where β_i is a Banach space of absolutely summable functions $(\epsilon_{:} i \in I)$.

$\nabla = (G_n \in \rho \times \epsilon : f_{G_n}$ is $(\mu \times \nu, \beta_{\epsilon_i} \times Y)$ - measurable).

$G_n = \bigcup_{k=1}^{n} E_k \times F_k$ where the union is disjoint and $E_k \times F_k \in \rho \times \epsilon$

for each k. Then

$< (\alpha \times \beta)(G_n), z^* > = \sum_{k=1}^{n} < \alpha(E_k)\beta(F_k), z^* >$

$= \sum_{i \in I} \sum_{k=1}^{n} <| \epsilon_i | (E_k)\beta(F_k), z^* >$

where $| \epsilon_i | (E_k) \in \beta_{\epsilon_i}$ for each $i \in I$ and $\beta(F_k) \in Y$ for $1 \leq k \leq n$

If $f_G = \chi_G$, then $f_{G_n}(s,t)$ is a $(\mu \times \nu, \beta_{\epsilon_i} \times Y)$ - valued function where

$G_n \in \rho \times \epsilon$ for each $(s,t) \in S \times T$. It follows that $G_n \in \nabla$ and $\rho \times \epsilon \subseteq \nabla$.

Therefore, $f_{G_n}(s,t)$ is a $(\mu \times \nu, \beta_{\epsilon_i} \times Y)$ - measurable.

Let $G'_n = ((x,y) :| f_{G_n}(x,y) - f_G(x,y) |) \geq 1 \setminus m$ for some n)

If $G'' = \bigcup_{k=1}^{\infty} G'_n$, then

$(G'')^c = \bigcap_{k=1}^{\infty} (G'_n)^c = \bigcap_{k=1}^{\infty} (((x,y) :| f_{G_n}(x,y) - f_G(x,y) |\geq 1 \setminus m)^c$

$= \bigcap_{k=1}^{\infty} (((x,y) :| f_{G_n}(x,y) - f_G(x,y) |< 1 \setminus m)$

where $(G'')^c$ is the complement of G''

Therefore, $(G''_n)^c \subset (((x,y) :| f_{G_n}(x,y) - f_G(x,y) |< 1 \setminus m)$

If $1 \setminus m < \lambda$ for $\lambda > 0$, then $\mid f_{G_n}(x, y) - f_G(x, y) \mid < \lambda$ for all $(x, y) \in (G'')^c$

where G'' is a null set. Therefore, $f_{G_n} \to f_G$ $(\mu \times \nu, \beta_{\epsilon_i} \times Y)$ - almost uniformly. Since f_{G_n} is $(\mu \times \nu, \beta_{\epsilon_i} \times Y)$ - a.e. bounded, then f_G is $(\mu \times \nu, \beta_{\epsilon_i} \times Y)$ bounded. It follows that f_G is $(\mu \times \nu, \beta_{\epsilon_i} \times Y)$ - measurable since it is the limit of a sequence $(f_{G_n})_{n=1}^{\infty}$ of $(\mu \times \nu, \beta_{\epsilon_i} \times Y)$ - measurable functions. Since f_G is bounded and measurable it implies that f_G is $(\mu \times \nu, \beta_{\epsilon_i} \times Y)$ - integrable.

Let $f_{G_n} \leq f_{G_n+1}$ a.e. for all $n \in \aleph$ and for a fixed $t \in T$. Then $< \int (\alpha_{f_{G_n(t)}}(E))\delta_{\beta(t)}, z^* > \leq m$ for $m > 0$ and $E \in \rho$. By monotone properties of a vector measure [10] , there exists an integrable function f_G such that $f_{G_n} \uparrow f_G$ and $LUB_n < \int (\alpha_{f_{G_n(t)}}(E))\delta_{\beta(t)}, z^* > = < \int (\alpha_{f_{G(t)}}(E))\delta_{\beta(t)}, z^* >$.

Since $G_n \uparrow G$ (hypothesis), it follows that

$$LUB_n \sum_{i \in I} <\mid \epsilon_i \mid \times \beta(G_n), z^* > = \sum_{i \in I} <\mid \epsilon_i \mid \times \beta(G), z^* >$$

4 Conclusion

The results obtained in this paper highlights the application of almost everywhere, measurability and boundedness properties to analyse absolute continuity of non-negative functions with values in a Hilbert space.

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Competing Interests

Author has declared that no competing interests exist.

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