Asian Research Journal of Mathematics

18(11): 342-350, 2022; Article no.ARJOM.92827

ISSN: 2456-477X

Asymptotic Expansions Related to the Wallis Ratio Based on the Bell Polynomials

Aimin Xu [a*]

[a] *Institute of Mathematics, Zhejiang Wanli University, Ningbo 315100, China.*

Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2022/v18i11608

Open Peer Review History:

This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/92827

Received: 20 August 2022

Accepted: 25 October 2022

Original Research Article

Published: 27 October 2022

Abstract

In this paper, we establish a new asymptotic expansion related to the Wallis ratio. By using the exponential Bell polynomials, we show that the coefficients of the asymptotic expansion can be recursively determined. In addition, an explicit expression for the coefficients is given. Our results improve and generalize the existing ones [1].

Keywords: Wallis ratio; Asymptotic expansion; Bell polynomial.

2010 Mathematics Subject Classification: 41A60; 33B15.

1 Introduction

Let $\mathbb{Z}_+$ be the set of all positive integers. For $n \in \mathbb{Z}_+$ the double factorial $n!!$ is defined by

$$n!! = \prod_{i=0}^{\lfloor (n-1)/2 \rfloor} (n - 2i), \tag{1.1}$$

where the floor function $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Corresponding author: E-mail: xuaimin1009@hotmail.com;

Using the double factorial, the Wallis ratio is usually defined as

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}, \tag{1.2}$$

where Γ is the classical Euler gamma function which is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad Re(z) > 0. \tag{1.3}$$

The study and applications of W_n have a long history and a lot of literature [2–6]. Recently, many authors paid attention to giving increasing better approximations for the Wallis ratio. For example, Guo, Xu and Qi proved in [7] that the double inequality

$$\sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n} < W_n \leq \frac{4}{3} \left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}, \tag{1.4}$$

where for $n \geq 2$ is valid and sharp in the sense that the constants $\sqrt{e/\pi}$ and $4/3$ are best possible. In their paper, Guo et al. also proposed an approximation formula for W_n as follows

$$W_n \sim \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}. \tag{1.5}$$

With the help of a lemma first proposed by Mortici [8], which plays a key role in many interesting works related to approximation of mathematical constants and special functions [9–25], Qi and Mortici [1] improved the double inequality (1.4) and the approximation formula (1.5). They provided a best approximation formula of the Wallis ratio:

$$W_n \sim \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2n}\right)^n \frac{1}{\sqrt{n}}, \quad n \to \infty, \tag{1.6}$$

and proved this formula is the best approximation of the form

$$W_n \sim \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n+a}}{n}, \quad n \to \infty, \tag{1.7}$$

where a is a real parameter. The approximation formula (1.6) can be viewed as an improvement of (1.5), because the approximation formula (1.5) is the special case $a = -1$ in (1.7). Qi and Mortici [1] further generalized the best approximation formula (1.6) as

$$W_n \sim \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2n}\right)^n \frac{1}{\sqrt{n}} \exp\left(\frac{a_2}{n^2} + \frac{a_3}{n^3} + \frac{a_4}{n^4} + \frac{a_5}{n^5} + \cdots\right), \quad n \to \infty, \tag{1.8}$$

where the a_k's are determined by an infinite triangular system

$$a_1 - \binom{k-1}{1} a_2 + \cdots + (-1)^k \binom{k-1}{k-2} a_{k-1} = \frac{1 + (-1)^k}{(k+1)2^{k+1}} - \frac{1}{k+1} + \frac{1}{2k}$$

which has a solution $a_1 = 0,\ a_2 = 1/24,\ a_3 = 1/48,\ a_4 = 1/160,\ a_5 = 1/960, \ldots$, and they improved the left-hand side of the double inequality (1.4) as

$$W_n > \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2n}\right)^n \frac{1}{\sqrt{n}} \exp\left(\frac{1}{24n^2} + \frac{1}{48n^3} + \frac{1}{160n^4} + \frac{1}{960n^5}\right). \tag{1.9}$$

Moreover, several other types of asymptotic expansions of the Wallis formula have been obtained by physical methods [26, 27] and probability integral method [28]. In [29, 30], various generalizations were derived.

Motivated by these interesting works, in this paper we will consider the following asymptotic expansion which is more general than (1.7):

$$W_n \sim \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{1}{\sqrt{n}}\left(1+\frac{c_1}{n}+\frac{c_2}{n^2}+\cdots\right)^{1/r}, \quad n\to\infty, \tag{1.10}$$

where r is a nonzero real number. One of the goals of this paper is to find an explicit expression for the coefficients $c_i (i = 1, 2, \ldots)$ based on the exponential complete Bell polynomials. Also, we will show that the c_i's can be recursively determined. Therefore, a series of the best approximation formulas of the Wallis ratio W_n including Formula (1.6) can be derived. In addition, an explicit expression of the coefficients $a_i (i = 2, 3, \ldots)$ in (1.8) will be derived. Our approach is based on the complete asymptotic expansion of $\ln\Gamma(x)$ and some basic facts from combinatorics. The work in this paper can be also considered as an application of the Bell polynomials in the asymptotic expansion of a type of special numbers.

2 Preliminaries

It is well known that Bell polynomials play very important role in enumerative combinatorics [31]. They are also important in our derivation, so we start from the definition of Bell polynomials and some properties of them. In the following, we are ready to introduce two kinds of Bell polynomials. One is of partial type and the other is of complete type.

The exponential partial Bell polynomials, named in honor of Bell [32], are the polynomials

$$\mathbb{B}_{n,k} := \mathbb{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \tag{2.1}$$

in an infinite number of variables $x_1, x_2, \ldots$, defined by the formal double series expansion:

$$\exp\left(u\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right) = \sum_{n,k\geq 0} \mathbb{B}_{n,k}\frac{t^n}{n!}u^k \tag{2.2}$$

or, what amounts to the same, by the series expansion:

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n\geq k} \mathbb{B}_{n,k}\frac{t^n}{n!}. \tag{2.3}$$

An alternative representation is

$$\mathbb{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{c_1!(1!)^{c_1} c_2!(2!)^{c_2}\cdots} x_1^{c_1} x_2^{c_2}\cdots, \tag{2.4}$$

where the summation takes place over all integers $c_1, c_2, \ldots \geq 0$, such that:

$$c_1 + 2c_2 + \cdots = n,$$
$$c_1 + c_2 + \cdots = k.$$

The first few cases of the exponential partial Bell polynomial are

$$\mathbb{B}_{1,1}(x_1) = x_1, \mathbb{B}_{2,1}(x_1, x_2) = x_2, \mathbb{B}_{2,2}(x_1) = x_1^2,$$
$$\mathbb{B}_{3,1}(x_1, x_2, x_3) = x_3, \mathbb{B}_{3,2}(x_1, x_2) = 3x_1x_2, \mathbb{B}_{3,3}(x_1) = x_1^3.$$

From (2.3) the exponential partial Bell polynomials can be recursively determined:

$$k\mathbb{B}_{n,k} = \sum_{l=k-1}^{n-1}\binom{n}{l} x_{n-l}\mathbb{B}_{l,k-1}. \tag{2.5}$$

Related to exponential partial Bell polynomials are exponential complete Bell polynomials $Y_n := Y_n(x_1, x_2, \ldots, x_n)$ defined by

$$\exp\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right) = 1 + \sum_{n\geq 1} Y_n \frac{t^n}{n!}, \tag{2.6}$$

in other words:

$$Y_n = \sum_{k=1}^{n} \mathbb{B}_{n,k}, \quad Y_0 := 1.$$

For an application of the Bell polynomials to the asymptotic expansion of the Somos recurrence constant, one is referred to [33].

Recall that B_i is the ith Bernoulli number defined by the power series expansion

$$\frac{x}{e^x - 1} = \sum_{i=0}^{\infty} B_i \frac{x^i}{i!} = 1 - \frac{x}{2} + \sum_{i=1}^{\infty} B_{2i} \frac{x^{2i}}{(2i)!}. \tag{2.7}$$

It is well known that $B_{2i+1} = 0$, for all $i \geq 1$, and the first few Bernoulli numbers are $B_1 = -1/2, B_2 = 1/6, B_4 = -1/30$ and $B_6 = 1/42$.

A formula approximating $\Gamma(x)$ for large values of x:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x,$$

known as Stirling's formula, is of special attraction. Many mathematicians are interested in improving such formulas in the form of an asymptotic series. It is now of general knowledge that the following Stirling series is (e.g., [34])

$$\Gamma(x+1) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}\right).$$

In fact, the above formula can be equivalently rewritten as the following lemma.

Lemma 2.1. *We have the complete asymptotic expansion of* $\ln\Gamma(x+1)$*:*

$$\ln\Gamma(x+1) \sim \left(x + \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln 2\pi + \sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}. \tag{2.8}$$

It is worth noting that since $B_{2i+1} = 0$, for all $i \geq 1$, Eq.(2.8) can be reformulated as an equivalent form

$$\ln\Gamma(x+1) \sim \left(x + \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln 2\pi + \sum_{j=1}^{\infty} \frac{B_{j+1}}{j(j+1)x^j}, \tag{2.9}$$

which will be used more conveniently in the next section.

3 Main Results

In order to determine the coefficients $c_i (i = 1, 2, \ldots)$ in (1.10), we firstly derive an explicit expression for the coefficients $a_i (i = 1, 2, \ldots)$ in the following asymptotic expansion due to Qi and Mortici [1].

Theorem 3.1. *As $n \to \infty$, we have*

$$W_n \sim \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{1}{\sqrt{n}} \exp\left(\sum_{i=1}^{\infty} \frac{a_i}{n^i}\right), \tag{3.1}$$

where

$$a_1 = 0, \quad a_i = \sum_{j=1}^{i-1} \frac{B_{j+1}}{j(j+1)} \binom{i-1}{i-j} \frac{1}{2^{i-j}}, \ i \geq 2. \tag{3.2}$$

Proof. From (1.2) and (2.9), it follows

$$\begin{aligned} \ln W_n &= -\frac{1}{2}\ln \pi + \ln \Gamma\left(n+\frac{1}{2}\right) - \ln \Gamma(n+1) \\ &\sim \frac{1}{2}(1-\ln \pi) + n \ln\left(1-\frac{1}{2n}\right) - \frac{1}{2}\ln n + \sum_{j=1}^{\infty} \frac{B_{j+1}}{j(j+1)(n-1/2)^j} - \sum_{j=1}^{\infty} \frac{B_{j+1}}{j(j+1)n^j}. \end{aligned}$$

Since

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{B_{j+1}}{j(j+1)(n-1/2)^j} &= \sum_{j=1}^{\infty} \frac{B_{j+1}}{j(j+1)n^j} \sum_{k=0}^{\infty} \binom{j+k-1}{k} \frac{1}{2^k} \frac{1}{n^k} \\ &= \sum_{i=1}^{\infty} \frac{1}{n^i} \sum_{j=1}^{i} \frac{B_{j+1}}{j(j+1)} \binom{i-1}{i-j} \frac{1}{2^{i-j}}, \end{aligned}$$

it is not difficult to obtain

$$\ln W_n \sim \frac{1}{2}(1-\ln \pi) + n \ln\left(1-\frac{1}{2n}\right) - \frac{1}{2}\ln n + \sum_{i=1}^{\infty} \frac{1}{n^i} \sum_{j=1}^{i-1} \frac{B_{j+1}}{j(j+1)} \binom{i-1}{i-j} \frac{1}{2^{i-j}}, \tag{3.3}$$

which is equivalent to (3.1) with

$$a_1 = 0, \quad a_i = \sum_{j=1}^{i-1} \frac{B_{j+1}}{j(j+1)} \binom{i-1}{i-j} \frac{1}{2^{i-j}}, \ i \geq 2.$$

Thus, the proof is complete. □

According to (3.2), the first few cases of the a_i's are

$$a_1 = 0, \quad a_2 = \frac{1}{24}, \quad a_3 = \frac{1}{48}, \quad a_4 = \frac{1}{160}, \quad a_5 = \frac{1}{960}.$$

See also Theorem 4.1 in [1]. Based on Theorem 3.1, we find a new asymptotic formula for W_n which generalizes the best approximation formula (1.6).

Theorem 3.2. *If r is a given nonzero real number, then the following asymptotic formula holds true:*

$$W_n \sim \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{1}{\sqrt{n}} \left(1+\frac{c_1}{n}+\frac{c_2}{n^2}+\cdots\right)^{1/r}, \quad n \to \infty, \tag{3.4}$$

where the coefficients c_i's are recursively determined by

$$\begin{aligned} c_1 &= 0, \\ c_i &= \frac{1}{i!}\sum_{j=2}^{i}(-1)^j (j-1)! \mathbb{B}_{i,j}(1!c_1, 2!c_2, \ldots, (i-j+1)!c_{i-j+1}) \\ &\quad + \sum_{j=1}^{i-1} \frac{rB_{j+1}}{j(j+1)} \binom{i-1}{i-j} \frac{1}{2^{i-j}}, \quad i \geq 2. \end{aligned} \tag{3.5}$$

Proof. From (3.3) and (3.4) it follows

$$\ln\left(1+\frac{c_1}{n}+\frac{c_2}{n^2}+\cdots\right)^{1/r}\sim\sum_{i=1}^{\infty}\frac{1}{n^i}\sum_{j=1}^{i-1}\frac{B_{j+1}}{j(j+1)}\binom{i-1}{i-j}\frac{1}{2^{i-j}}. \tag{3.6}$$

It is well known that

$$\ln\left(1+\sum_{j=1}^{\infty}\frac{c_j}{n^j}\right)=\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i}\left(\sum_{j=1}^{\infty}\frac{c_j}{n^j}\right)^i. \tag{3.7}$$

Therefore, by (3.6) and (3.7) we have

$$\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i}\left(\sum_{j=1}^{\infty}\frac{c_j}{n^j}\right)^i\sim\sum_{i=1}^{\infty}\frac{1}{n^i}\sum_{j=1}^{i-1}\frac{rB_{j+1}}{j(j+1)}\binom{i-1}{i-j}\frac{1}{2^{i-j}}.$$

The definition of the exponential partial Bell polynomials yields

$$\begin{aligned}&\sum_{i=1}^{\infty}\frac{1}{i!}\frac{1}{n^i}\sum_{j=1}^{i}(-1)^{j-1}(j-1)!\mathbb{B}_{i,j}(1!c_1,2!c_2,\ldots,(i-j+1)!c_{i-j+1})\\ &\sim\sum_{i=1}^{\infty}\frac{1}{n^i}\sum_{j=1}^{i-1}\frac{rB_{j+1}}{j(j+1)}\binom{i-1}{i-j}\frac{1}{2^{i-j}}.\end{aligned}$$

Equating the coefficients of $1/n^i$ on both sides and noting that $\mathbb{B}_{i,1}(1!c_1,2!c_2,\ldots,i!c_i)=i!c_i$, we derive (3.5). ☐

Notice that c_i does not appear in the right-hand side of (3.5). Notice also both the Bell polynomials $\mathbb{B}_{i,j}$'s and the Bernoulli numbers B_j's can be recursively calculated. This means that (3.5) can be computed by using symbolic software such as Maple or Mathematica. Having previously computed $c_1,c_2,\ldots,c_{i-1}$, we can then compute c_i using (3.5). In fact, the c_i's can also be explicitly given in terms of the exponential complete Bell polynomials.

Theorem 3.3. *If r is a given nonzero real number, then the following asymptotic formula holds true:*

$$W_n\sim\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n\frac{1}{\sqrt{n}}\left(1+\frac{c_1}{n}+\frac{c_2}{n^2}+\cdots\right)^{1/r},\quad n\to\infty, \tag{3.8}$$

where the coefficients c_i's can be explicitly given by

$$\begin{aligned}c_1&=0,\\ c_i&=\frac{1}{i!}Y_i(b_1,b_2,\ldots,b_i),\ i\ge 2.\end{aligned} \tag{3.9}$$

Here

$$b_1=0,\quad b_i=i!\sum_{j=1}^{i-1}\frac{rB_{j+1}}{j(j+1)}\binom{i-1}{i-j}\frac{1}{2^{i-j}},\ i\ge 2.$$

Proof. From (3.6) it follows

$$1+\sum_{i=1}^{\infty}\frac{c_i}{n^i}\sim\exp\left(\sum_{i=1}^{\infty}\frac{1}{n^i}\sum_{j=1}^{i-1}\frac{rB_{j+1}}{j(j+1)}\binom{i-1}{i-j}\frac{1}{2^{i-j}}\right).$$

According to (2.6) we have

$$\exp\left(\sum_{i=1}^{\infty}\frac{1}{n^i}\sum_{j=1}^{i-1}\frac{rB_{j+1}}{j(j+1)}\binom{i-1}{i-j}\frac{1}{2^{i-j}}\right)=1+\sum_{i\geq 1}Y_i(b_1,b_2,\ldots,b_i)\frac{1}{i!}\frac{1}{n^i},$$

where

$$b_1=0,\quad b_i=i!\sum_{j=1}^{i-1}\frac{rB_{j+1}}{j(j+1)}\binom{i-1}{i-j}\frac{1}{2^{i-j}},\ i\geq 2.$$

Equating the coefficients of $1/n^i$ yields (3.9). □

Finally, by virtue of Theorems 3.2 and 3.3 we give the first few cases of the coefficient c_i to end this section.

$$c_1=0,\quad c_2=\frac{r}{24},\quad c_3=\frac{r}{48},\quad c_4=\frac{r^2}{1152}+\frac{r}{160},\quad c_5=\frac{r^2}{1152}+\frac{r}{960}.$$

Acknowledgements

This work was supported by the Ningbo Natural Science Foundation under grant 2021J179.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Qi F, Mortici C. Some best approximation formulas and inequalities for the Wallis ratio. Appl. Math. Comput. 2015;253:363-368.

[2] Chen C-P, Qi F. The best bounds in Wallis' inequality. Proc. Amer. Math. Soc. 2005;133(2):397-401.

[3] Guo B-N, Qi F. A class of completely monotonic functions involving divided differences of the psi and tri-gamma functions and some applications. J. Korean Math. Soc. 2011;48(3):655-667.

[4] Qi F. Integral representations and complete monotonicity related to the remainder of Burnside's formula for the gamma function. J. Comput. Appl. Math. 2014;268:155-167.

[5] Qi F, Cui L-H, Xu S-L. Some inequalities constructed by Tchebysheff's integral inequality. Math. Inequal. Appl. 1999;2(4):517-528.

[6] Qi F, Luo Q-M. Complete monotonicity of a function involving the gamma function and applications. Period. Math. Hungar. 2014;69(2):159-169.

[7] Guo S, Xu J-G, Qi F. Some exact constants for the approximation of the quantity in the Wallis formula. J. Inequal. Appl. 2013;2013(67):1-7.

[8] Mortici C. Product approximations via asymptotic integration. Amer. Math. Monthly. 2010;117(5):434-441.

[9] Chen C-P, Srivastava HM. New representations for the Lugo and Euler-Mascheroni constants. II. Appl. Math. Lett. 2012;25(3):333-338.

[10] Furdui O. A class of fractional part integrals and zeta function values. Integral Transforms Spec. Funct. 2013;24(6):485-490.

[11] Laforgia A, Natalini P. On the asymptotic expansion of a ratio of gamma functions. J. Math. Anal. Appl. 2012;389(2):833-837.

[12] Lin L. Further refinements of Gurland's formula for π. J. Inequal. Appl. 2013;2013(48):1-11.

[13] Lu D-W. A new quicker sequence convergent to Euler's constant. J. Number Theory. 2014;136:320-329.

[14] Lu D-W. A generated approximation related to Burnside's formula. J. Number Theory. 2014;136:414-422.

[15] Lu D-W. Some new convergent sequences and inequalities of Euler's constant. J. Math. Anal. Appl. 2014;419:541-552.

[16] Lu D-W, Fang J-H, Ma C-X. A general asymptotic formula of the gamma function based on the Burnside's formula. J. Number Theory. 2014;145:317-328.

[17] Lu D-W, Mortici C. Some new quicker approximations of Glaisher-Kinkelin's and Bendersky-Adamchik's constants. J. Number Theory. 2014;144:340-352.

[18] Lu D-W, Wang X-G. A generated approximation related to Gosper's formula and Ramanujan's formula. J. Math. Anal. Appl. 2013;406(1):287-292.

[19] Mahmoud M, Alghamdi MA, Agarwal RP. New upper bounds of $n!$. J. Inequal. Appl. 2012;2012(27):1-9.

[20] Mansour M, Obaid MA. On a new Stirling's series. Ars Combin. 2012;107:411-418.

[21] Mortici C. A new Stirling series as continued fraction. Numer. Algorithms. 2011;56(1):17-26.

[22] Mortici C. A quicker convergence toward the γ constant with the logarithm term involving the constant e. Carpathian J. Math. 2010;26(1):86-91.

[23] Mortici C. Fast convergences towards Euler-Mascheroni constant. Comput. Appl. Math. 2010;29(3):479-491.

[24] Mortici C. New improvements of the Stirling formula. Appl. Math. Comput. 2010;217(2):699-704.

[25] Mortici C. Very accurate estimates of the polygamma functions. Asymptot. Anal. 2010;68(3):125-134.

[26] Chashchina OI, Silagadze ZK. On the quantum mechanical derivation of the Wallis formula for π. Phys. Lett. A. 2017;381:2593.

[27] Cortese I, García JA. Wallis formula from the harmonic oscillator. arXiv:1709.01214v2.

[28] Schatz JR. Wallis formula and the probability integral. arXiv:2209.12760v1.

[29] Hu S, Kim M-S. Mizuno-type result and Wallis formula. arXiv:2109.01477v2.

[30] Hu S, Kim M-S. A generalization of Wallis formula. arXiv:2201.09674v1.

[31] Comtet L. Advanced combinatorics, the art of finite and infinite expansions. D. Reidel Publishing Co., Dordrecht; 1974.

[32] Bell ET. Exponential polynomials. Ann. of Math. 1934;35:258-277.

[33] Xu A. Asymptotic expansion related to the generalized Somos recurrence constant. International Journal of Number Theory. 2019;15(10):2043-2055.

[34] Abramowitz M, Stegun IA. Handbook of mathematical functions with formulas, graphs, and mathematical tables. Dover Publications, New York; 1972.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle5.com/review-history/92827