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A Common Fixed-point Theorem of Mappings on S-metric Spaces

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

Fixed Point Theory is developed during the years by generalizing various types of contractions or changing the axioms of metric spaces. The S-metric spaces extend the concept of metric spaces since the determination of axioms of S-metric are more general. In this paper, we prove a common fixed-point theorem of two mappings in S-metric space. The theorem uses an implicit contractive condition, based on functions of class F_6. With the change of the form of the functions of class F_6, we obtain different results for common fixed points of two functions.

Keywords: S-metric space; common fixed point; contractive condition; function of class F_6; weakly compatible mappings.

1 Introduction

Fixed Point Theory was studied for the first time in 1922 by Stephan Banach [1]. In its publication, Banach [1] proved the existence and the uniqueness of a fixed point for a contractive mappings in metric space. It had an importance and role in Functional Analysis, Differential Equations, Physics for its applicable sides. During the years, many authors ([2-5]) have worked on this direction defining new contractions and new spaces by generalizing the results of Banach [1].

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In 2012, S. Sedghi et al [1] introduced the notions of S-metric space. The authors in references ([6-8]) showed some properties of S-metric spaces and proved some fixed points results in these spaces.

Recently, Özgür et al [5] and Prudhvi [4] [6] obtained some results in S-metric spaces.

The results related to the fixed points in various spaces are given considering functions that satisfy explicit contractive conditions. On the other hand, lately some results are verified taking in consideration more general contractive conditions given in implicit way from Popa and Berinde [8] [9] [10], Duraj and Hoxha [11] [12].

In 2021, Patriciu [7] proved one general fixed point theorem for mappings which satisfy a cyclical contractive condition using a new type of implicit relation.

In this paper, we prove some common fixed point results for two of weakly compatible mappings in S-metric space using a new implicit relation. The presented results extend various known fixed points theorems given in references [8-12].

2 Preliminiries

In this section we present some results related to S -metric spaces taken by various authors.

Definition 2.1: ([1] [2]) Let X be a non-empty set. An S -metric on X is a function $S: X^3 \to \mathbb{R}^+$ such that satisfies the following conditions:

(S_1): $S(x, y, z) = 0$ if and only if $x = y = z$.

(S_2): $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

The pair (X, S) is called an S -metric space.

Example 2.2: [7] Let $X = \mathbb{R}$ be the set of real numbers and $S(x, y, z) = |x - z| + |y - z|$. Then $S(x, y, z)$ is an S -metric on $\mathbb{R}$ which is named the usual S-metric on X.

Example 2.3: Let $X = \mathbb{R}$ be the set of real numbers and $S(x, y, z) = \begin{cases} 0, & x = y = z \\ max\{x, y, z\}, & otherwise \end{cases}$. Then (X, S) is an S -metric space.

Lemma 2.4: [1] If (X, S) is an S-metric space on a nonempty set X, then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.

Definition 2.5:

a) A sequence $\{x_n\}$ in (X, S) is called convergent to x, (denoted $\lim_{n\to\infty} x_n = x$ or $x_n \to x$) if $S(x_n, x_n, x) \to 0$ as $n \to \infty$.
b) A sequence $\{x_n\}$ in (X, S) is called a Cauchy sequence, if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.
c) (X, S) is called complete if every Cauchy sequence is convergent in it.

Lemma 2.6: ([6,8][1] [3]) Let (X, S) be a S-metric space. If $x_n \to x$ and $y_n \to y$ then $S(x_n, x_n, y_n) \to S(x, x, y)$.

Lemma 2.7: ([6,8][1] [2]) Let (X, S) be a S-metric space. If $x_n \to x$. Then $\lim\limits_{n\to\infty} x_n$ is unique.

Definition 2.8: Let T and g be a self-mappings of a nonempty set X. A point $x \in X$ is said to be a common fixed point of T and g if $x = Tx = gx$.

Definition 2.9: [13] Let (X, S) and (X', S') be two S -metric spaces and let $f: (X, S) \to (X', S')$ be a function. Then f is said to be continuous at a point $a \in X$ if for every sequence $\{x_n\}$ in X, completes $S(x_n, x_n, f(a)) \to a$, implies $S'(f(x_n), f(x_n), f(a)) \to 0$. A function f is continuous at X and if and only if it is continuous for every $a \in X$.

3 Main Results

In this section, we prove some new results for implicit function of class $\boldsymbol{\mathcal{F}_6}$ in a complete $\boldsymbol{S}$-metric space.

Let us consider the class $\mathcal{F}_6$of all the functions lower semi-continuous $F: \mathbb{R}_+^6 \to \mathbb{R}$ that satisfy the following conditions:

$(\boldsymbol{F_1})$: F is non decreasing in t;

$(\boldsymbol{F_2})$:There exists $k\epsilon[0,1/2)$ such that for all $u, v \geq 0$ $F(u, v, v, u, 0, 2u + v) \leq 0$ implies $u \leq kv$.

$(\boldsymbol{F_3})$: $F(t, t, 0, 0, t, t) > 0, \ \forall t > 0$.

In Example 3.1 we show various functions of class $\mathcal{F}_6$.

Example 3.1:[7]

a) $F(t_1, t_2, \dots, t_6) = t_1 - at_2 - bt_3 - b\, max\{t_3, t_4, t_5, t_6\}$ where $a, b \geq 0$ and $a + 3b < \frac{1}{2}$.
b) $F(t_1, t_2, \dots, t_6) = t_1 - at_2 - bt_3 - ct_4 - dmax\{t_5, t_6\}$ where $a, b, c, d \geq 0$ and $a + b + c + 3d < \frac{1}{2}$.
c) $F(t_1, t_2, \dots, t_6) = t_1 - at_2 - dmax\{t_3, t_4\} - bt_5 - ct_6$, where $a, b, c, d \geq 0$ and $a + 3c + d < \frac{1}{2}$.
d) $F(t_1, t_2, \dots, t_6) = 1 - amax\{t_4 + t_5, t_3 + t_6\} - bt_2$, where $a, b \geq 0$ and $4a + b < \frac{1}{2}$.
e) $F(t_1, t_2, \dots, t_6) = t_1 - k\, max\{t_2, t_3, t_4, t_5, t_6\}$ where $k\epsilon[0, \frac{1}{3})$.
f) $F(t_1, t_2, \dots, t_6) = t_1 - k\, max\{t_2, t_3, t_4, \frac{t_5 + t_6}{3}\}$ where $k\epsilon[0, \frac{1}{2})$.

In following we prove a new theorem for a couple of functions that satisfy an implicit relation using a map of class $\mathcal{F}_6$.

Theorem 3.2: Let (X, S) be a complete S-metric space and let $T, g: X \to X$ when T is continuous in X. If the inequality:

$$F(S(Tx, gy, gy), S(x, y, y), \ S(x, Tx, Tx), S(y, gy, gy), S(y, Tx, Tx), S(x, gy, gy)) \leq 0 \tag{1}$$

holds for all $x, y\epsilon X$ and $F\epsilon\mathcal{F}_6$, then T and g have a common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. Then $x_1 = Tx_0$ and $x_2 = gx_1$.

By the condition (1) of the Theorem 3.2 for $x = x_0, y = x_1$, we have

$$F(S(Tx_0, gx_1, gx_1), S(x_0, x_1, x_1), S(x_0, Tx_0, Tx_0), S(x_1, gx_1, gx_1),$$
$$S(x_1, Tx_0, Tx_0), S(x_0, gx_1, gx_1)) \leq 0$$

$$F(S(x_1, x_2, x_2), S(x_0, x_1, x_1), S(x_0, x_1, x_1), S(x_1, x_2, x_2), S(x_1, x_1, x_1), S(x_0, x_2, x_2)) \leq 0$$

Considering (S_2) and Lemma 2.4, we have

$$S(x_0, x_2, x_2) = S(x_2, x_2, x_0) \leq S(x_2, x_2, x_1) + S(x_2, x_2, x_1) + S(x_0, x_0, x_1) = 2 \quad S(x_2, x_2, x_1) + S(x_0, x_1, x_1).$$

Using (S_1), we have $S(x_1, x_1, x_1) = 0$.

Consequently,

$$F(S(x, x_2, x_2), S(x_0, x_1, x_1), S(x_0, x_1, x_1), S(x_1, x_2, x_2), 0, 2S(x_1, x_2, x_2) + S(x_0, x_1, x_1)) \leq 0.$$

By (F_2) exists $k\epsilon[0,\frac{1}{2})$ such that $S(x_1,x_2,x_2) \leq kS(x_0,x_1,x_1)$.

Continuing this process, we can define inductively the sequence $\{x_n\}$ as follows:

$x_{2n+1} = Tx_{2n}$ and $x_{2n} = gx_{2n-1}$.

Replacing $x = x_{2n}$ and $y = x_{2n+1}$ in (1) we have:

$$F(S(Tx_{2n}, gx_{2n+1}, gx_{n+1}), S(x_{2n}, x_{2n+1}, x_{2n+1}), S(x_{2n}, Tx_{2n}, Tx_{2n}),$$
$$S(x_{2n+1}, gx_{2n+1}, gx_{2n+1}), S(x_{2n+1}, Tx_{2n}, Tx_{2n}), S(x_{2n}, gx_{2n+1}, gx_{2n+1})) \leq 0.$$

$$F(S(x_{2n+1}, x_{2n+2}, x_{2n+2}), S(x_{2n}, x_{2n+1}, x_{2n+1}), S(x_{2n}, x_{2n+1}, x_{2n+1}),$$
$$S(x_{2n+1}, x_{2n+2}, x_{2n+2}), S(x_{2n+1}, x_{2n+1}, x_{2n+1})S(x_{2n}, x_{2n+2}, x_{2n+2})) \leq 0.$$

Using (S_2) and Lemma 2.4, we have:

$$\begin{aligned} S(x_{2n}, x_{2n+2}, x_{2n+2}) &= S(x_{2n+2}, x_{2n+2}, x_{2n}) \leq S(x_{2n+2}, x_{2n+2}, x_{2n+1}) + S(x_{2n+2}, x_{2n+2}, x_{2n+1}) + S(x_{2n}, x_{2n+1}, x_{2n+1} \\ &= 2S(x_{2n+2}, x_{2n+2}, x_{2n+1}) + S(x_{2n}, x_{2n+1}, x_{2n+1}) \end{aligned}$$

$$S(x_{2n+1}, x_{2n+1}, x_{2n+1}) = 0.$$

So,

$$F(S(x_{2n+1}, x_{2n+2}, x_{2n+2}); S(x_{2n}, x_{2n+1}, x_{2n+1});\ S(x_{2n}, x_{2n+1}, x_{2n+1}); S(x_{2n+1}, x_{2n+2}, x_{2n+2}), 0,$$
$$2S(x_{2n+2}, x_{2n+2}, x_{2n+1}) + S(x_{2n}, x_{2n+1}, x_{2n+1})) \leq 0.$$

Due to (F_2), there exists $k\epsilon[0,\frac{1}{2})$ such that $S(x_{2n+1}, x_{2n+2}, x_{2n+2}) \leq kS(x_{2n}, x_{2n+1}, x_{2n+1})$.

Similarly, we obtain $S(x_{2n}, x_{2n+1}, x_{2n+1}) \leq kS(x_{2n-1}, x_{2n}, x_{2n})$.

Hence, we have

$$S(x_{2n+1}, x_{2n+2}, x_{2n+2}) \leq kS(x_{2n}, x_{2n+1}, x_{2n+1}) \leq k^2 S(x_{2n-1}, x_{2n}, x_{2n}) \leq \ldots \leq k^{2n} S(x_0, x_1, x_1),$$

where $n\epsilon\mathbb{N}$; (2)

or $S(x_n, x_{n+1}, x_{n+1}) \leq k^n S(x_0, x_1, x_1)$ where $n\epsilon\mathbb{N}$. (3)

Let us show that the sequence $\{x_n\}$ is Cauchy. Using (S_2) we have:

$$\begin{aligned} &S(x_n, x_n, x_m) = S(x_m, x_m, x_n) \leq S(x_m, x_m, x_{n+1}) + S(x_m, x_m, x_{n+1}) + S(x_n, x_{n+1}, x_{n+1}) \leq \\ &\leq 2\, S(x_m, x_m, x_{n+1}) +\ S(x_n, x_{n+1}, x_{n+1}) \\ &\leq 2S(x_m, x_m, x_{n+1}) + k^n S(x_0, x_1, x_1) \\ &\leq 2[S(x_m, x_m, x_{n+2}) + S(x_m, x_m, x_{n+2}) + S(x_{n+1}, x_{n+1}, x_{n+2})] + k^n S(x_0, x_1, x_1) \\ &\leq 2^2 S(x_m, x_m, x_{n+2}) + 2k^{n+1}\, S(x_0, x_1, x_1) + k^n S(x_0, x_1, x_1) \leq \cdots \\ &\leq 2^{m-n-1} k^{m-1} S(x_0, x_1, x_1) + 2^{m-n-2}\, k^{m-2} S(x_0, x_1, x_1) + \cdots + 2k^{n+1} S(x_0, x_1, x_1) + k^n S(x_0, x_1, x_1) \end{aligned}$$

$$= [(2k)^{m-n-1} + (2k)^{m-n-2} + 2k + 1]\, k^n S(x_0, x_1, x_1)$$

$$= \frac{[(2k)^{m-n} - 1]}{2k-1} k^n\, S(x_0, x_1, x_1).$$

Taking limit when $n, m \to \infty$ and since $k\epsilon[0,\frac{1}{2})$, we have that $\lim_{m,n\to\infty} S(x_n, x_n, x_m) = 0$, hence the sequence $\{x_n\}$ is Cauchy [13-19].

Since (X, S) is complete, it follows that $\{x_n\}$ is convergent to a point $u \epsilon X$. So the $\lim_{n\to\infty} S(x_n, x_n, u) = 0$ (4).

We have to prove that u is a common fixed point of T and g.

For $x = x_{2n}$ and $y = u$ by (1) we obtain:

$$F(S(Tx_{2n}, gu, gu), S(x_{2n}, u, u), S(x_{2n}, Tx_{2n}, Tx_{2n}), S(u, gu, gu), S(u, Tx_{2n}, Tx_{2n}),$$
$$S(x_{2n}, gu, gu)) \leq 0.$$

$$F(S(x_{2n+1}, gu, gu), S(x_{2n}, u, u), S(x_{2n}, x_{2n+1}, x_{2n+1}),$$
$$S(u, gu, gu), S(u, x_{2n+1}, x_{2n+1}), S(x_{2n}, gu, gu)) \leq 0.$$

By taking the limit when $n \to \infty$ and using (3) and (4), we have:

$$F(S(u, gu, gu), S(u, u, u), S(u, u, u), S(u, gu, gu), S(u, u, u), S(u, gu, gu)) \leq 0$$

$$F(S(u, gu, gu); 0;\ 0; S(u, gu, gu), 0, S(u, gu, gu)) \leq 0$$

Using (F_1), we get

$$F(S(u, gu, gu); 0;\ 0; S(u, gu, gu), 0, 2S(u, gu, gu)) \leq 0$$

Considering (F_2), we obtain $S(u, gu, gu)) \leq k \cdot 0 = 0$ and $S(u, gu, gu)) = 0$ and by (S_1) it follows that $u = gu$. Hence u is a fixed point of g.

Since T is continuous form (4), we have that $\lim_{n\to\infty} S(x_{2n}, x_{2n}, u) = 0$ and we get $\lim_{n\to\infty} S(Tx_{2n}, Tx_{2n}, Tu) = 0.$

But $0 = \lim\limits_{n\to\infty} S(Tx_{2n}, Tx_{2n}, Tu) = \lim_{n\to\infty} S(x_{2n+1}, x_{2n+1}, Tu) = S(u, u, Tu)$

From (S_1) we have $Tu = u$ and u is a fixed point of T. Hence $u = gu = Tu$ is common fixed point of T and g.

Remark 3.3. If we take, $T \equiv g$ then the above theorem reduces to Theorem 3 of [7].

Remark 3.4. By taking different functions $F \epsilon \mathcal{F}_6$ (see Example 3.1) in Theorem 3.2, we obtain various results for the existence and uniqueness of a common fixed point for two functions in S-metric spaces.

4 Conclusion

In this article, there are studied some fixed point results in S-metric spaces. The highlight of this paper is Theorem 3.2. in which there is proved the existence and uniqueness of a common fixed points of two functions in S-metric space which satisfy an implicit contraction using maps of class $\mathcal{F}_6$. Furthermore, in Theorem 3.2 there is taken only one of the functions continuous, concretely the function T while for the function g the continuity does not need.

In addition, by taking different maps from Example 3.1 and replacing in Theorem 3.2, there are obtained various results about the existence and uniqueness of a common point for two functions in S-metric space.

Competing Interests

Authors have declared that no competing interests exist.

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