

Research Article

Analysis of Bianchi Type V Holographic Dark Energy Models in General Relativity and Lyra's Geometry

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In this paper, we analyze anisotropic and homogeneous Bianchi type V spacetime in the presence of dark matter and holographic dark energy model components in the framework of general relativity and Lyra's geometry. The solutions of differential equation fields have been obtained by considering two specific cases, namely, the expansion scalar (θ) in the model is proportional to the shear scalar (σ) and the average scale factor taken as hybrid expansion form. The solutions for field equations are obtained in general relativity and Lyra's geometry. The energy density of dark matter in both natures was obtained and compared so that the energy density of dark matter in general relativity is slightly different from the energy density of dark matter in Lyra's geometry. A similar behavior occurred in case of pressure and EoS parameter of holographic dark energy model in respective frameworks. Also, it is concluded that the physical parameters such as the average Hubble parameter, spatial volume, anisotropy parameter, expansion scalar, and shear scalar are the same in both frameworks. Moreover, it is observed that the gauge function $\beta(t)$ is a decreasing function of cosmic time in Lyra's geometry, and for late times, the gauge function ($\beta(t)$) converges to zero and Lyra's geometry reduces to general relativity in all respects. Finally, we conclude that our models are a close resemblance to the Λ CDM cosmological model in late times and consistent with the recent observations of cosmological data.

1. Introduction

The relatively recent discovery of the present acceleration of our universe is strictly proved by the astronomical observation of type Ia supernova [1, 2], galaxy redshift survey [3], and cosmic microwave background radiation (CMBR) data [4, 5] which convincingly suggest that the rate of expansion of our universe is positive, i.e., we live in an accelerating expanding universe. This fact initiated a great number of theoretical hypothesis and inspired researchers to a diversity of different explanations of such an unusual behavior of the universe. The most surprising and counterintuitive result coming from these observations is the fact that only $\approx 4\%$ of the total energy density of the universe is in the form of baryonic matter, $\approx 24\%$ is nonbaryonic matter, and almost $\approx 72\%$ is of completely unknown component with negative pressure. In literature, the component with negative pressure is named as

dark energy (DE) that produces repulsive force which gives rise to the current accelerating expansion of the universe.

The most fundamental principle of quantum gravity, the holographic principle (HP), states that all of the information contained in a volume of space can be represented as a hologram, which corresponds to a theory locating on the boundary of that space that may play an important role in solving the DE problem. After applying the HP to the DE problem, one of the present authors [6] proposed a new DE model, called holographic dark energy (HDE) model. In this model, the dark energy density ρ_Λ only relies on two physical quantities on the boundary of the universe: one is the reduced Planck mass $M_p \equiv \sqrt{1/8\pi G}$, where G is the gravitational constant; another is the cosmological length scale L , which is chosen as the future event horizon of the universe [6].

The HDE model is the first theoretical model of DE inspired by the HP and is in good agreement with the current

cosmological observations at the same time. This makes HDE a very competitive candidate of DE. Granda and Oliveros [7] proposed a holographic density of the form $\rho_\Lambda \approx \alpha H^2 + \gamma \dot{H}$, where H is the Hubble parameter and α, γ are constants which must satisfy the restrictions imposed by the current observational data. They showed that this new model of dark energy represents the accelerated expansion of the universe and is consistent with the current observational data. Granda and Oliveros [8] have also studied the correspondence between the quintessence, tachyon, k -essence, and dilation dark energy models with this holographic dark energy model in the flat FRW universe. Chattopadhyay and Debnath [9], Farajollahi et al. [10], Debnath [11], Malekjani [12], and Sarkar [13, 14] are some of the researchers who have investigated several aspects of holographic dark energy. Recently, Kiran et al. [15, 16] have studied minimally interacting dark energy models in some scalar-tensor theories. Rao et al. [17] and Adhav et al. [18] have discussed interacting dark matter and HDE in Bianchi type V universe. Rao et al. [19] have discussed the five-dimensional FRW holographic dark energy in the Brans-Dicke theory.

In recent years, our knowledge of cosmology has improved remarkably by various experimental and theoretical results. Einstein [20] introduced his general theory of relativity in which gravitation is described in terms of geometry of spacetime. Motivated by it, Einstein geometrized other physical fields in general relativity. One of the first attempts in this direction was made by Weyl [21] who proposed a more general theory in which gravitation and electromagnetism are also described geometrically. However, this theory was never considered seriously as it was based on the nonintegrability of length transfer. Later, Lyra [22] suggested a modification of Riemannian geometry by introducing a gauge function which removes the nonintegrability condition of the length of a vector under parallel transport. This modified Riemannian geometry is known as Lyra's geometry. Subsequently, Sen [23] proposed a new scalar tensor theory of gravitation. They constructed an analog of Einstein's field equations based on Lyra's geometry which is in normal gauge. He found that the static model with finite density in Lyra's manifold is similar to the static model in Einstein's general relativity.

Halford [24] has shown that the constant displacement vector field in Lyra's geometry plays the role of cosmological constant Λ in general relativity. He has also shown that the scalar-tensor analysis based on Lyra's geometry suggests the same effects, within observational limits, as in Einstein's theory [20]. Katore et al. [25] studied the Einstein-Rosen bulk viscous cosmological model and zero-mass scalar field in Lyra's geometry. Ghate and Sontakke [26], Asgar and Ansari [27], and Das and Sharma [28] studied the Bianchi type V cosmological models in Lyra's geometry with dark energy, in the presence of bulk viscous string; Sahu et al. [29] studied the Bianchi type III cosmological model in Lyra's geometry, and Katore and Kapse [30] studied dynamics of Bianchi type VI₀ holographic dark energy models in general relativity and Lyra's geometry.

Motivated by the abovementioned investigations, we have considered the homogeneous Bianchi type V holographic dark energy cosmological models in general relativity

and Lyra's geometry. This paper is outlined as follows. In Section 2, we discussed the metric and the field equations in general relativity. In Section 3, we have obtained the solutions of field equations. Some physical properties of the models have been studied in Section 4. In Section 5, we have obtained the field equations in Lyra's geometry; its solutions and physical properties are studied under Sections 5.1 and 5.2, respectively. Finally, the conclusions are summarized in Section 6.

2. Metric and Field Equations

We consider the spatially homogeneous and anisotropic Bianchi type V spacetime as

$$ds^2 = dt^2 - A^2 dx^2 - B^2 e^{-2m_1 x} dx^2 - C^2 e^{-2m_1 x} dx^2, \quad (1)$$

where m_1 is an arbitrary constant and A, B, C are functions of cosmic time t only.

The Einstein field equations are given by

$$R_{ij} - \frac{1}{2} R g_{ij} = -\frac{8\pi G}{c^4} T_{ij}, \quad (2)$$

where R_{ij} , R , g_{ij} , G , and c are the Ricci tensor, Ricci scalar, metric tensor, Newton's gravitational constant, and speed of light, respectively. Here, we consider $8\pi G = c = 1$. T_{ij} is the energy momentum tensor which is expressed as the sum of the energy momentum tensors of dark matter ($T_{(m)ij}$) and the holographic dark energy ($T_{(\Lambda)ij}$).

The energy momentum tensor of holographic dark energy ($T_{(\Lambda)ij}$) of the source with anisotropic pressures along different spatial directions has the form

$$T_{(\Lambda)ij} = \text{diag} [\rho_\Lambda, -p_x, -p_y, -p_z] = \text{diag} [1, -\omega_x, -\omega_y, -\omega_z] \rho_\Lambda, \quad (3)$$

where ρ_Λ and $p_i (i = x, y, z)$ are the energy density of holographic dark energy (HDE) and pressures of the HDE in the three different directions of the universe. A relation between ρ_Λ and p_i is given by equation of state (EoS) parameter as $p_i = \omega_i \rho_\Lambda$, and ω_i are equation of state parameters along directions of x, y , and z .

Here, we suppose that the universe is filled with dark matter and dark energy whose energy momentum tensors can be written as

$$T_{(m)ij} = \text{diag} [1, 0, 0, 0] \rho_m, \quad (4)$$

$$T_{(\Lambda)ij} = \text{diag} [1, -\omega_\Lambda, -(\omega_\Lambda + \delta_y), -(\omega_\Lambda + \delta_z)] \rho_\Lambda, \quad (5)$$

where ρ_m and ρ_Λ are the energy densities of dark matter and holographic dark energy (HDE), respectively, and p_Λ is the pressure of the HDE. For the sake of simplicity, we choose $\omega_x = \omega_\Lambda$, $\omega_y = \omega_\Lambda + \delta_y$, $\omega_z = \omega_\Lambda + \delta_z$ and the skewness parameters δ_y and δ_z are the deviations from the equation of state parameter ω_Λ on y and z directions, respectively.

The physical parameters are defined as follows.

The mean Hubble parameter (H) is given by

$$H = \frac{\dot{a}}{a} = \frac{1}{3} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right), \quad (6)$$

where the overhead dot denotes differentiation with respect to the cosmic time t .

The average scale factor (a) and spatial volume (V) of the Bianchi V spacetime are defined by

$$V = a^3 = ABC. \quad (7)$$

The scalar expansion (θ), shear scalar (σ), anisotropy parameter (A_m), and deceleration parameter (q) are defined as

$$\begin{aligned} \theta &= 3H = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}, \\ \sigma^2 &= \frac{1}{2} \left(\sum_{i=1}^3 H_i^2 - 3H^2 \right), \\ A_m &= \frac{1}{3} \sum_{i=1}^3 \left(\frac{H_i - H}{H} \right)^2, \\ q &= -1 + \frac{d}{dt} \left(\frac{1}{H} \right). \end{aligned} \quad (8)$$

As suggested by Granda and Oliveros [8], the energy density of HDE with IR cut-off is given by

$$\rho_\Lambda = 3(\alpha H^2 + \gamma \dot{H}), \quad (9)$$

with $M_p^{-2} = 8\pi G = 1$, where M_p is the reduced Planck mass and α and γ are the dimensionless parameters, which must satisfy the restrictions imposed by the current observational data. The energy conservation equation $(T_{(m)ij} + T_{(\Lambda)ij})_{;j} = 0$ yields

$$\begin{aligned} \dot{\rho}_m + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \rho_m + \dot{\rho}_\Lambda + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) (1 + \omega_\Lambda) \rho_\Lambda \\ + \left(\delta_y \frac{\dot{B}}{B} + \delta_z \frac{\dot{C}}{C} \right) \rho_\Lambda = 0. \end{aligned} \quad (10)$$

By adopting comoving coordinates, the field equation (2) for the metric (1) using the energy-momentum tensors (5) yields the following equations:

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{m_1^2}{A^2} = -\omega_\Lambda \rho_\Lambda, \quad (11)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{m_1^2}{A^2} = -(\omega_\Lambda + \delta_y) \rho_\Lambda, \quad (12)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{m_1^2}{A^2} = -(\omega_\Lambda + \delta_z) \rho_\Lambda, \quad (13)$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{3m_1^2}{A^2} = (\rho_\Lambda + \rho_m), \quad (14)$$

$$\frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0, \quad (15)$$

where the overhead dot denotes ordinary differentiation with respect to cosmic time t .

Integrating equation (15) and absorbing the constant of integration into B or C , we obtain

$$A^2 = BC. \quad (16)$$

3. Solution of Field Equations

We can observe that the field equations (11)–(15) are a system of five independent equations with the eight unknown parameters A , B , C , ω_Λ , ρ_m , ρ_Λ , δ_y , and δ_z . In order to solve the system completely, we need more additional conditions to obtain explicit solution of the system.

So first, we consider the hybrid expansion law in which the average scale factor is an increasing function of cosmic time t as follows:

$$a = (t^n e^t)^{1/k}, \quad (17)$$

where $k > 0$ and $n \geq 0$ are constants. This type of ansatz for the scale factor has already been considered by Akarsu et al. [31], Moraes et al. [32], Moraes and Sahoo [33], Ram and Chandel [34], Pradhan and Amirhashchi [35], Pradhan et al. [36], and Santhi et al. [37]. The relation (17) gives the exponential law when $n = 0$. This is a combination of exponential and power law, which is termed as hybrid expansion law. Also, this relation yields a time-dependent deceleration parameter which describes the transition of the universe from the early decelerating phase to the current accelerating phase. Thus, our choice of scale factor is physically acceptable.

Secondly, we assume that the expansion scalar (θ) in the model is proportional to shear scalar (σ) as considered by Thorne [38] and Collins et al. [39], which leads to

$$A = B^l, \quad l > 0 (l \neq 1), \quad (18)$$

where A and B are the metric potentials.

From equations (16), (17), and (18), we obtain

$$A = (t^n e^t)^{1/k}, \quad (19)$$

$$B = (t^n e^t)^{1/kl}, \quad (20)$$

$$C = (t^n e^t)^{(2l-1)/kl}. \quad (21)$$

The metric (1) can now be written as

$$ds^2 = dt^2 - (t^n e^t)^{2/k} dx^2 - (t^n e^t)^{2/kl} e^{-2m_1 x} dx^2 - (t^n e^t)^{(2(2l-1))/kl} e^{-2m_1 x} dx^2. \quad (22)$$

4. Some Physical and Geometrical Features of the Model

The physical parameters such as the Hubble parameter (H), the anisotropic parameter (A_m), the shear scalar (σ^2), the expansion scalar (θ), and the spatial volume (V) of model (22), which are of cosmological importance, are, respectively, given by

$$H = \frac{1}{k} \left(\frac{n}{t} + 1 \right), \quad (23)$$

$$A_m = \frac{2}{3} \left(\frac{1}{l} - 1 \right)^2, \quad (24)$$

$$\sigma^2 = \frac{1}{k^2} \left(\frac{n}{t} + 1 \right)^2 \left(\frac{1}{l} - 1 \right)^2, \quad (25)$$

$$\theta = \frac{3}{k} \left(\frac{n}{t} + 1 \right), \quad (26)$$

$$V = (t^n e^t)^{3/k}. \quad (27)$$

The directional Hubble parameters are as follows:

$$\begin{aligned} H_x &= \frac{\dot{A}}{A} = \frac{1}{k} \left(\frac{n}{t} + 1 \right), \\ H_y &= \frac{\dot{B}}{B} = \frac{1}{kl} \left(\frac{n}{t} + 1 \right), \\ H_z &= \frac{\dot{C}}{C} = \frac{2l-1}{kl} \left(\frac{n}{t} + 1 \right). \end{aligned} \quad (28)$$

From $a(t) = 1/(z+1)$, with z being the redshift and the present scale factor $a_0 = 1$. Using (17), we obtain the following redshift

$$z = -1 + (t^n e^t)^{-1/k}. \quad (29)$$

It is evident from the above result in equation (27) that the spatial volume is zero at $t = 0$. But we observe that as cosmic time $t \rightarrow \infty$, the spatial volume expands to infinite. Therefore, the model starts evolving at $t = 0$ and expands with cosmic time t . The mean anisotropy parameter is constant and different from zero for $l \neq 1$. From equation (24), it is observed that the mean anisotropy parameter of the present model becomes zero for $l = 1$ and the anisotropy of the universe vanishes. But the universe is anisotropic throughout the evolution except for $l = 1$. Other dynamical physical parameters such as expansion scalar (θ), shear scalar (σ^2), and Hubble parameter (H) diverge as cosmic time t approaches to zero. These dynamical physical parameters

are decreasing functions as cosmic time t increases. Hence, the model (22) has a big bang type of initial singularity.

The deceleration parameter (q) is obtained to be

$$q = -\frac{a\ddot{a}}{a^2} = -1 + \frac{nk}{t^2((n/t) + 1)^2}, \quad (30)$$

which is a constant value for late time throughout the evolution of the universe that means $q \rightarrow -1$ as $t \rightarrow \infty$. From equation (30), the universe will expand with decelerated rate for $q > 0$, i.e., $t < \sqrt{nk} - n$, accelerated rate for $q < 0$, i.e., $\sqrt{nk} - n < t$, and marginal inflation for $q = 0$, i.e., $t = \sqrt{nk} - n$. One can explicitly observe the dependence of deceleration parameter (q) on the constant parameters n and k . Thus, we can obtain a decelerated or accelerated expansion of the universe depending on the suitable choices of these parameters. According to the recent observations of type Ia supernova (SNe Ia) [2, 40, 41, 42], the present universe is accelerating and the value of deceleration parameter is in the range $-1 < q < 0$. Thus, the deceleration parameter of our model (22) is consistent with the recent astronomical observations.

Following Granda and Oliveros [8] and Sarkar [13] and using equation (23) in equation (9), the energy density of HDE with the IR cut-off is given by

$$\rho_\Lambda = 3(\alpha H^2 + \gamma \dot{H}) = 3 \left(\frac{\alpha}{k^2} \left(\frac{n}{t} + 1 \right)^2 - \frac{\gamma n}{kt^2} \right). \quad (31)$$

Using equation (31) in equation (14), the energy density of matter is given by

$$\rho_m = \left(\frac{2l^2 + 2l - 1}{(kl)^2} - \frac{3\alpha}{k^2} \right) \left(\frac{n}{t} + 1 \right)^2 + \frac{3\gamma n}{kt^2} - \frac{3m_1^2}{(t^n e^t)^{2/k}}. \quad (32)$$

From equation (11), the pressure of HDE with the IR cut-off is given by

$$p_\Lambda = \left[-\frac{1}{(kl)^2} \left(\frac{n}{t} + 1 \right)^2 (4l^2 - 2l + 1) + \frac{2n}{kt^2} + \frac{m_1^2}{(t^n e^t)^{2/k}} \right]. \quad (33)$$

Here, from (31), we observe that the holographic dark energy density (ρ_Λ) decreases as cosmic time t increases. The energy density of dark matter is infinite at cosmic time $t = 0$, and the pressure of holographic dark energy is a decreasing function of cosmic time. From (33), we concluded that the pressure of holographic dark energy has negative value as cosmic time $t \rightarrow \infty$.

Using equations (31) and (33), the EoS parameter of HDE is obtained as

$$\omega_\Lambda = \frac{\left[-(1/(kl)^2)((n/t) + 1)^2(4l^2 - 2l + 1) + (2n/kt^2) + (m_1^2/(t^n e^t)^{2/k}) \right]}{3((\alpha/k^2)((n/t) + 1)^2 - (\gamma n/kt^2))}. \quad (34)$$

From equations (11), (12), and (13), the skewness parameter along y direction is

$$\sigma_y = \frac{(1/k^2)((n/t) + 1)^2((1/l) - 1) + (n/kt)(1 - (1/l))}{3((\alpha/k^2)((n/t) + 1)^2 - (\gamma n/kt^2))}. \quad (35)$$

From equations (11), (12), and (13), the skewness parameter along z direction is

$$\sigma_z = \frac{(3/k^2)((n/t) + 1)^2(1 - (1/l)) + (n/kt)((1/l) - 1)}{3((\alpha/k^2)((n/t) + 1)^2 - (\gamma n/kt^2))}. \quad (36)$$

Thus, the metric (22) together with equations (31)–(36) constitutes a Bianchi type V HDE cosmological model in general relativity with hybrid expansion of the universe. The behavior of the energy densities depends on the values of constants α , γ , k , and n . Moreover, the energy density of dark matter depends on constants m_1 and l . From equations (31) and (32), we observe that the energy densities of matter and HDE are decreasing functions of time. Also, we have seen that from equation (34) it is also observed that the pressure of HDE is a decreasing function of cosmic time t . The obtained EoS parameter of HDE is time varying and it is evolving with negative sign for late times which may be attributed to the current acceleration of the expansion the universe [43]. The EoS parameter of the HDE also behaves like quintessence or phantom region based on choices of constants.

The coincidence parameter is

$$\bar{r} = \frac{\rho_\Lambda}{\rho_m} = \frac{3((\alpha/k^2)((n/t) + 1)^2 - (\gamma n/kt^2))}{((2l^2 + 2l - 1)/(kl^2) - (3\alpha/k^2))((n/t) + 1)^2 + (3\gamma n/kt^2) - (3m_1^2/(t^n e^t)^{2/k})}. \quad (37)$$

It is observed that coincidence parameter \bar{r} at very early stage of evolution varies, but after some finite time, it converges to a constant value and remains constant throughout the evolution, thereby avoiding the coincidence problem (unlike Λ CDM).

The energy densities of parameters of dark matter (Ω_m) and HDE (Ω_Λ) are as follows:

$$\Omega_m = \frac{1}{3} \left(\frac{2l - 1}{l^2} + (2 - 3\alpha) \right) + \left(\frac{k\gamma n}{t^2} - \frac{k^2 m_1^2}{(t^n e^t)^{2/k}} \right) \left(\frac{n}{t} + 1 \right)^{-2}, \quad (38)$$

$$\Omega_\Lambda = \alpha - \frac{\gamma n k}{t^2} \left(\frac{n}{t} + 1 \right)^{-2}. \quad (39)$$

Using equations (38) and (39), we get the sum of overall energy density parameter as

$$\Omega = \Omega_m + \Omega_\Lambda = \frac{1}{3} \left(\frac{2l - 1}{l^2} + 2 \right) - \left(\frac{k^2 m_1^2}{(t^n e^t)^{2/k}} \right) \left(\frac{n}{t} + 1 \right)^{-2}. \quad (40)$$

From equation (40), one can observe that the overall sum of the energy density parameter approaches $(1/3)((2l - 1)/l^2 + 2)$ as $t \rightarrow \infty$. Moreover, it can be concluded that for $l = 1$, this model predicts that the sum of overall energy density parameter becomes 1. Figure 1 shows the nature of total density parameter as cosmic time increases. As it was illustrated in Figure 1, total density parameter (Ω) tends to 1 as cosmic time increases, which is physically acceptable based on the recent cosmological data.

Figure 2 shows the accelerated phase of the universe ($q \approx -1$) for some small values of redshift. This illustrates that the findings are in accordance with the observational values of the current cosmological data of accelerated universe. Figure 3 indicates the energy density of dark matter versus cosmic time t for constants $l = 1.5$, $n = 2$, $\alpha = 0.5$, $k = 0.5$, $m_1 = 0.5$, and $\gamma = 0.6$. From Figure 3, it can be seen that at early stage of the evolution, energy density dark matter dominates and at late times it approaches zero. This supports the recent observations of the cosmological data.

Figure 4 indicates the plot of pressure of holographic dark energy versus cosmic time t for constants $l = 1.5$, $n = 2$, $k = 0.5$, and $m_1 = 0.5$. As it is shown in Figure 4, the negative pressure of holographic dark energy increases as cosmic time t increases. The negative pressure indicates the accelerated phase of the expansion of the universe. Figure 5 shows the equation of state parameter of dark energy with respect to cosmic time. As it was shown in Figure 5, the dark energy of equation of state lies $-1/3 < w_{de} < -1$ which implies the region of quintessence. Figure 1 shows total density parameter versus cosmic time. As it was shown in Figure 1, the total density parameter (Ω) approaches to 1 at late time which exactly matches with the Λ CDM model of recent cosmological data.

In order to discriminate among the various DE models, Sahni et al. [44] and Alam et al. [45] introduced a new geometrical diagnostic pair for DE, which is known as statefinder pair and denoted as (r, s) . The statefinder pair probes the dynamic expansion of the universe through higher derivatives of the scale factor. Hence, it is a geometrical diagnosis in the sense that it depends on the scale factor which describes the spacetime.

The statefinder pair is defined as

$$r = \frac{\ddot{a}}{aH^3}, \quad (41)$$

$$s = \frac{r - 1}{3(q - (1/2))}.$$

Substituting the required values from equations (17), (23), and (30) into (41), we get

$$r = 1 - \frac{3nk}{t^2((n/t) + 1)^2} + \frac{2nk^2}{t^3((n/t) + 1)^3}, \quad (42)$$

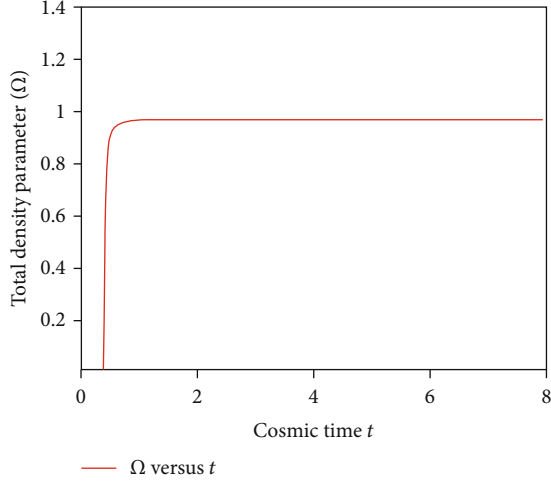


FIGURE 1: The plot of total density parameter (Ω) versus cosmic time t for constants $l = 1.5$, $n = 2$, $k = 0.5$, and $m_1 = 0.5$.

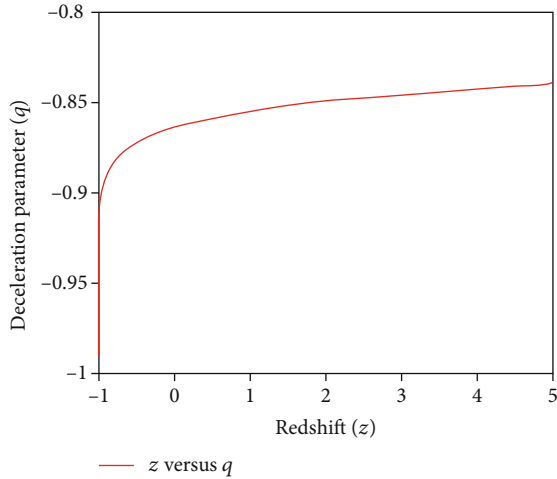


FIGURE 2: Deceleration parameter evolution in redshift for constants $l = 1.5$, $n = 2$, and $k = 0.5$.

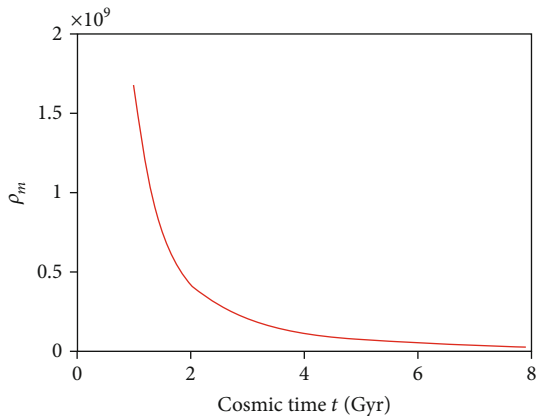


FIGURE 3: The plot of energy density of dark matter versus cosmic time t for constants $l = 1.5$, $n = 2$, $\alpha = 0.5$, $k = 0.5$, $m_1 = 0.5$, and $\gamma = 0.6$.

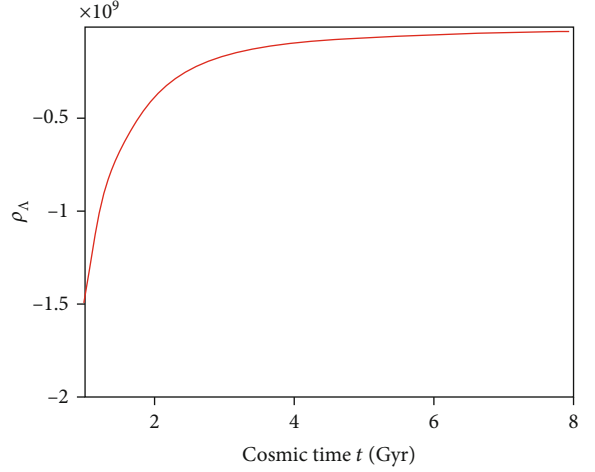


FIGURE 4: The plot of pressure of holographic dark energy versus cosmic time t for constants $l = 1.5$, $n = 2$, $k = 0.5$, and $m_1 = 0.5$.

$$s = \frac{-(3nk/(t^2((n/t) + 1)^2)) + (2nk^2/(t^3((n/t) + 1)^3))}{3((nk/(t^2((n/t) + 1)^2)) - (3/2))}. \quad (43)$$

From equations (42) and (43), it can be observed that the statefinder parameters are constant whose values depend on k and n in late times. Sahni et al. [44] and Alam et al. [45] have observed that the Lambda cold dark matter (Λ CDM) model and the standard cold dark matter (SCDM) model have fixed point values of statefinder parameter $(r, s) = (1, 0)$ and $(r, s) = (1, 1)$, respectively. Figure 6 shows $r - s$ statefinder parameters. As it was shown in Figure 6, our model approaches to $(r, s) \rightarrow (1, 0)$ as cosmic time t approaches to infinity which would clearly show the Λ CDM model.

5. Lyra's Geometry and Field Equations

Lyra [22] suggested a modification of Riemannian geometry by introducing a gauge function which is a metrical concept in Weyl [21] geometry in the geometrical structureless manifold. It is a generalization of Weyl's geometry which is removing the defect of nonintegrability of length transfer that has been discussed. The Einstein modified field equation in normal gauge for Lyra's manifold obtained by Sen [23] is given by

$$R_{ij} - \frac{1}{2}Rg_{ij} + \frac{3}{2}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi_k\phi^k = -\left(T_{(m)ij} + T_{(\Lambda)ij}\right), \quad (44)$$

where ϕ_i is the displacement vector defined as $\phi_i = (0, 0, 0, \beta(t))$.

In a comoving coordinate system, the modified Einstein field equation (44) for Bianchi type V spacetime with the help of (5) is

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{m_1^2}{A^2} + \frac{3}{4}\beta^2 = -\omega_\Lambda\rho_\Lambda, \quad (45)$$

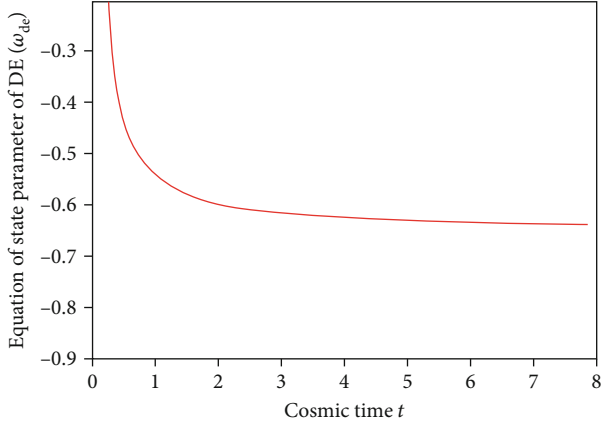


FIGURE 5: The plot of equation of state parameter of dark energy (ω_{de}) versus cosmic time t for constants $l = 1.5$, $n = 2$, $k = 0.5$, $m_1 = 0.5$, $\alpha = 0.5$, and $\gamma = 0.6$.

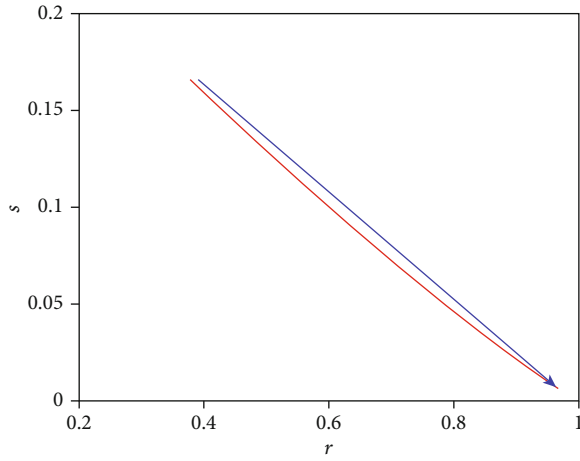


FIGURE 6: The plot of $r - s$ for constants $l = 1.5$, $n = 2$, $k = 0.5$, and $m_1 = 0.5$.

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{m_1^2}{A^2} + \frac{3}{4}\beta^2 = -(\omega_\Lambda + \delta_y)\rho_\Lambda, \quad (46)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{m_1^2}{A^2} + \frac{3}{4}\beta^2 = -(\omega_\Lambda + \delta_z)\rho_\Lambda, \quad (47)$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{3m_1^2}{A^2} - \frac{3}{4}\beta^2 = (\rho_\Lambda + \rho_m), \quad (48)$$

$$\frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0, \quad (49)$$

where the overhead dot denotes derivative with respect to cosmic time t .

The conservation of the right-hand side of equation (44) leads to

$$\left(R_i^j - \frac{1}{2}Rg_i^j\right)_{;j} + \frac{3}{2}(\phi_i\phi^j)_{;j} - \frac{3}{4}(g_i^j\phi_k\phi^k)_{;j} = 0. \quad (50)$$

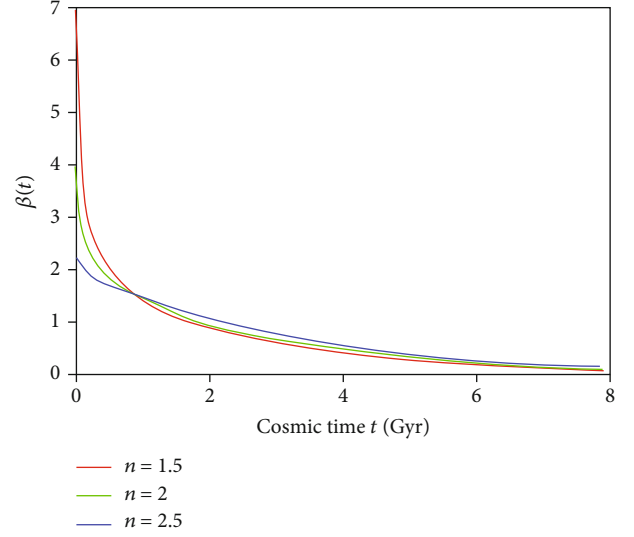


FIGURE 7: The plot of gauge function $\beta(t)$ versus cosmic time t for $n = 1.5, 2$, and 2.5 and $k = 0.5$.

Equation (50) is reduced to

$$\begin{aligned} \frac{3}{2}\phi_i \left[\frac{\partial\phi^j}{\partial x^j} + \phi^l \Gamma_{ij}^j \right] + \frac{3}{2}\phi^j \left[\frac{\partial\phi_i}{\partial x^j} - \phi_l \Gamma_{ij}^l \right] - \frac{3}{4}g_i^j \phi_k \left[\frac{\partial\phi^k}{\partial x^j} + \phi^l \Gamma_{ij}^k \right] \\ - \frac{3}{4}g_i^j \phi^k \left[\frac{\partial\phi_k}{\partial x^j} - \phi_l \Gamma_{kj}^l \right] = 0, \end{aligned} \quad (51)$$

leading to

$$\frac{3}{2}\beta\dot{\beta} + \frac{3}{2}\beta^2 \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 0. \quad (52)$$

Integrating equation (49), we obtain

$$A^2 = BC, \quad (53)$$

where the constant of integration is absorbed in B or C .

5.1. Solutions of Field Equations. Using equations (6) and (17) in (52) and integrating, we obtain the gauge function β

$$\beta(t) = \frac{c_1}{(t^n e^t)^{3/k}}, \quad (54)$$

where c_1 is a constant of integration.

Using the same conditions above in (9) and (21), we have solved the field equations (45)–(49). From equation (54), we observe that at early time, the gauge function β tends to infinity in this limit and also it tends to zero as cosmic time $t \rightarrow \infty$. It is large in the beginning and decreases fast with the evolution of the universe. Hence, $\beta(t)$ is a decreasing function of cosmic time t . Thus, the model has singularity at $t = 0$.

Figure 7 indicates the plot of β versus cosmic time t for the constant values $n = 1.5, 2$, and 2.5 and $k = 0.5$. It clearly

shows that the behavior of β is as decreasing function of cosmic time t . It can be concluded that as n values increase, the values of $\beta(t)$ decrease more with cosmic time increases. The physical parameters such as Hubble parameter (H), anisotropic parameter (A_m), shear scalar (σ^2), expansion scalar (θ), spatial volume (V), and deceleration parameter (q) are found to be the same as in the above case of general relativity. Thus, we get the model (22).

5.2. Some Physical Properties of the Model. Using equations (9) and (21), the solutions of the field equations (45)–(49) are obtained. Therefore, the skewness parameters, energy density of dark matter (ρ_m), pressure of HDE (p_Λ), and EoS parameter of HDE (ω_Λ) are given below as follows.

Here, from equations (45) and (46), we obtain

$$\begin{aligned} \delta_y &= - \left(\frac{(\ddot{A}/A) - (\ddot{B}/B) + (\dot{A}\dot{C}/AC) - (\dot{B}\dot{C}/BC)}{\rho_\Lambda} \right) \\ &= \frac{(1/k^2)((n/t) + 1)^2((1/l) - 1) + (n/kt)(1 - (1/l))}{3((\alpha/k^2)((n/t) + 1)^2 - (\gamma n/kt^2))}. \end{aligned} \quad (55)$$

From equation (54), it is noted that the gauge function β is a decreasing function of time which is corroborated with Halford [46] as well as with the recent observations [2, 47, 48][1, 40] leading to the conclusion that Λ (cosmological

Also, from equations (45) and (47), we get

$$\begin{aligned} \delta_z &= - \left(\frac{(\ddot{A}/A) - (\dot{C}/C) + (\dot{A}\dot{B}/AB) - (\dot{B}\dot{C}/BC)}{\rho_\Lambda} \right) \\ &= \frac{(3/k^2)((n/t) + 1)^2(1 - (1/l)) + (n/kt)((1/l) - 1)}{3((\alpha/k^2)((n/t) + 1)^2 - (\gamma n/kt^2))}. \end{aligned} \quad (56)$$

From equation (48), the energy density of dark matter is

$$\begin{aligned} \rho_m &= \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{3m_1^2}{A^2} - \frac{3}{4}\beta^2 - \rho_\Lambda \\ &= \left(\frac{2l^2 + 2l - 1}{(kl)^2} - \frac{3\alpha}{k^2} \right) \left(\frac{n}{t} + 1 \right)^2 + \frac{3\gamma n}{kt^2} - \frac{3m_1^2}{(t^n e^t)^{2/k}} - \frac{3c_1^2}{4(t^n e^t)^{6/k}}, \end{aligned} \quad (57)$$

$$\begin{aligned} p_\Lambda &= - \left(\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{m_1^2}{A^2} + \frac{3}{4}\beta^2 \right) \\ &= \left[- \frac{1}{(kl)^2} \left(\frac{n}{t} + 1 \right)^2 (4l^2 - 2l + 1) + \frac{2n}{kt^2} + \frac{m_1^2}{(t^n e^t)^{2/k}} - \frac{3c_1^2}{4(t^n e^t)^{6/k}} \right]. \end{aligned} \quad (58)$$

From equation (45), we can get the EoS parameter as EoS parameter of holographic dark energy in general relativity are slightly different from those in Lyra's geometry. It can be seen that the behavior of EoS parameter strongly depends on parameters of the model. However, by choosing appropri-

$$\omega_\Lambda = \frac{- (1/(kl)^2)((n/t) + 1)^2(4l^2 - 2l + 1) + (2n/kt^2) + (m_1^2/(t^n e^t)^{2/k}) - (3c_1^2/4(t^n e^t)^{6/k})}{3((\alpha/k^2)((n/t) + 1)^2 - (\gamma n/kt^2))}. \quad (59)$$

constant) is a decreasing function of cosmic time t . From equation (57), it is observed that the energy density of dark matter is a decreasing function of time. From equation (58), it is also observed that the pressure of HDE is a decreasing function of time. The energy density of holographic dark energy in Lyra's geometry is similar to the energy density of HDE in general relativity. Similarly, the skewness parameters along y and z directions in general relativity and Lyra's geometry will be the same. But the dark energy density of dark matter, the pressure of holographic dark energy, and the

From equation (60), it is observed that coincidence parameter \bar{r} at the very early stage of evolution varies, but after some finite time, it converges to a constant value and

ate values for α , m_1 , n , k , and γ , we can obtain ω_Λ in quintessence or in phantom phase which is consistent with the recent observations of cosmological data. The EoS parameter of holographic dark energy is obtained as time varying, and it is evolved with negative sign for late times and its range is in good agreement with large-scale structure of the universe data.

The coincidence parameter is

remains constant throughout the evolution, thereby avoiding the coincidence problem (like Λ CDM).

$$\bar{r} = \frac{\rho_\Lambda}{\rho_m} = \frac{3((\alpha/k^2)((n/t) + 1)^2 - (\gamma n/kt^2))}{((2l^2 + 2l - 1)/(kl)^2) - (3\alpha/k^2)((n/t) + 1)^2 + (3\gamma n/kt^2) - (3m_1^2/(t^n e^t)^{2/k}) - (3c_1^2/4(t^n e^t)^{6/k})}. \quad (60)$$

The energy density parameter of dark matter in Lyra's geometry is given by

$$\Omega_m = \frac{\rho_m}{3H^2} = \frac{1}{3} \left(\frac{2l-1}{l^2} + (2-3\alpha) \right) + \left(\frac{k\gamma n}{t^2} - \frac{k^2 m_1^2}{(t^n e^t)^{2/k}} - \frac{k^2 c_1^2}{4(t^n e^t)^{6/k}} \right) \left(\frac{n}{t} + 1 \right)^{-2}. \quad (61)$$

Using (39) and (61), the overall sum of density parameter (Ω) is obtained as

$$\Omega = \frac{1}{3} \left(\frac{2l-1}{l^2} + 2 \right) - \left(\frac{k^2 m_1^2}{(t^n e^t)^{2/k}} + \frac{k^2 c_1^2}{4(t^n e^t)^{6/k}} \right) \left(\frac{n}{t} + 1 \right)^{-2}. \quad (62)$$

The overall sum of the energy density parameter approaches $(1/3)((2l-1)/l^2 + 2)$ as cosmic time $t \rightarrow \infty$. It converges to 1 as $t \rightarrow \infty$ for the constant $l=1$, and it holds for $l > 1$ which is physically acceptable. Hence, it indicates that the universe is very close to critical density or $\Omega = 1$. Thus, the gauge function $\beta(t)$ is large in the start of the model but decays continuously during its evolution. The energy density and the pressure tend to small positive and negative constants, respectively, for the large values of cosmic time t . The negative pressure corresponds to the accelerated phase of the universe. The gauge function $\beta(t)$ becomes zero for large time t . In this case, the solutions reduce to general relativistic one.

6. Conclusions

In this paper, we have studied anisotropic and homogeneous Bianchi type V universe filled with dark matter and holographic dark energy in the framework of general relativity and Lyra's geometry. The solutions for field equations of cosmological models are obtained in general relativity and Lyra's geometry by using two specific cases: firstly by taking the expansion scalar (θ) in the model is proportional to the shear scalar (σ) as considered by Thorne [38] and Collins et al. [39] and secondly by using the average scale factor as hybrid expansion form (combination of the power law and exponential form). We have investigated the nature of deceleration parameter versus redshift as shown in Figure 2 which reveals that the accelerated nature of the universe maintained for small values of redshift. Figure 3 indicates that the energy density of dark matter versus cosmic time t is a decreasing function. It is constructed based on arbitrary constants constructed domain. Here, it can be concluded that at early stage of the evolution, energy density of dark matter dominates and at late times it approaches zero. This supports the recent observations of cosmological data of the universe. Figure 4 shows the pressure of HDE versus cosmic time t . It shows that the pressure of HDE negatively increases with time. The overall density parameter has been matched with Λ CDM as shown in Figure 1 which indicates that $\Omega \approx 1$. The energy density of dark matter, the pressure of HDE, the EoS parameter of HDE, the coincidence parameter, and overall density parameter in Lyra's geometry slightly differ by the

term β from general relativity framework. We have observed that the gauge function $\beta(t)$ is a decreasing function of cosmic time t in Lyra's geometry. The gauge function $\beta(t)$ is large in the beginning and reduces fast with the evolution of the universe. It is found that for late cosmic times the gauge function $\beta \rightarrow 0$ and Lyra's geometry tends to general relativity in all respects. The present model of the gauge function $\beta(t)$ is infinite at the initial singularity. From Figure 7, we see that β tends to zero as $t \rightarrow \infty$. Therefore, the concept of the Lyra manifold is meaningful for finite time, but does not remain for very large time. But we have observed that the dynamical parameters: the average Hubble parameter, spatial volume, anisotropy parameter, expansion scalar and shear scalar, are the same in both frameworks. In each case, the cosmological models approach to anisotropic parameter for large value of cosmic time t . For $l=1$, anisotropic model vanishes. These models represent a shearing, nonrotating, and expanding universe, which approaches anisotropy for large value of time t . The EoS parameter of the HDE also behaves like quintessence region based on the selected choice of the constants and parameters which is shown in Figure 5. From geometrical diagnostic pair for DE, it can be seen that $(r, s) \rightarrow (1, 0)$ as cosmic time t approaches to infinity and it is clearly illustrated in Figure 6 which is consistent with the Λ CDM model. The present model is consistent with the recent observations of the present day of accelerating universe.

Data Availability

The data used to support the findings of this study are included within the article and are cited at relevant places within the text as references.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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