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Existence and uniqueness of solutions for stochastic urban-population growth model

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Urban-population growth model has attracted attention over the last few decades due to its usefulness in representing population dynamics, virus dynamics, and epidemics. Researchers have included stochastic perturbation in the urban-population growth model to improve the model, attempting to capture the random nature of real-time dynamics. When doing so, researchers have presented conditions to ensure that the corresponding stochastic solution is both positive and unique (in probability). This paper advances that knowledge by showing that the stochastic diffusion constant can be both positive and negative—previous results in the literature have required that such a constant be positive only. A numerical simulation illustrates the paper's findings.

KEYWORDS

stochastic system, urban-population growth model, positive systems, uniqueness of solution, stochastic evolution equation

1. Introduction

The Hokkaido prefecture, in Japan, had experienced a severe economic crisis that hit hard the social and living situation in that region [1]. Searching for better opportunities, individuals living in the countryside of that region started moving to the Sapporo city, capital of the Hokkaido prefecture. This migration led to an unbalance in Hokkaido's population distribution and created all kinds of social problems [1].

In an attempt to understand the population dynamics within the Hokkaido prefecture, a group of researchers has applied real-time population data to the so-called *dynamic self-organization theory* [see [1]]. This theory was first developed by Nicolis and Prigogine [2] in the classical monograph; this theory tries to explain a phenomenon in which a system organizes itself through internal and external interactions within the local population. The key idea is to let the interactions between two local populations be driven by a system of two deterministic differential equations. In formal terms, the deterministic differential equations are [e.g., [1]]

$$\begin{aligned} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= k_1x_1(t)\big(N_1 - x_1(t) - \beta x_2(t)\big) - d_1x_1(t), \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= k_2x_2(t)\big(N_2 - x_2(t) - \beta x_1(t)\big) - d_2x_2(t), \end{aligned} \tag{1}$$

where $x_1(t)$ and $x_2(t)$ are positive terms that represent the population on the first and second region, respectively, and the constants k_i, N_i, and d_i, $i = 1, 2$, are positive and known constants. The constant $\beta > 0$ sets the interdependence level between the two regions [cf., [1]].

The authors of [1] have considered a linearization of the model (1) at the point $(\bar{x}_1, \bar{x}_2) = (N_1 - d_1/k_1, 0)$, finding conditions for local stability, valid only around the point $(\bar{x}_1, \bar{x}_2)$. Another study has shown that (1) has four different equilibrium points and that $x_i(t)$, $i = 1, 2$, converges to one of these points as t increases to infinity [see [3]]. When reaching a convergence point, the two regions' population enters into equilibrium [cf., [3]]. In other words, the asymptotic stability of the system (1) is completely characterized by the authors of [3]. However, as pointed out in [[4], Chapter 5], the model in (1) remains incomplete because (1) does not account for random fluctuations that usually drive the behavior of population dynamics.

Researchers have become interested in stochastic differential equations for modeling population dynamics because these equations have proved to be useful in a variety of applications, such as in epidemics [5–7], fish population [8], phytoplankton concentration [9], HIV (virus) dynamics [10, 11], dengue [12], and tumor cell growth [13]. Researchers have even considered the stochastic version of the deterministic two-region population dynamics shown in (1) [e.g., [14–16]]. What researchers have proposed is, in fact, a stochastic model that simply adds the term $\sigma_i x_i(t)\mathrm{d}B_i(t)$, $i = 1, 2$, in (1), where $B_i(t)$ denotes the standard unidimensional Brownian motion. The resulting stochastic differential equation is then studied in a way that the corresponding solution is unique [14–16]. To ensure uniqueness, the authors of [14–16] require that $\sigma_i > 0$, $i = 1, 2$; however, as we show in this paper, that condition is unnecessary—we show that σ_i, $i = 1, 2$, can be both positive and negative. This finding represents the main contribution of this paper.

The main contribution of this paper is to show the conditions that guarantee the stochastic urban-population growth model have a unique, positive solution. What this paper advances with respect to the previous results from the literature [e.g., [14–16]] is that this paper shows uniqueness and positiveness of solution without the classical assumption that $\sigma_i > 0$, $i = 1, 2$. This paper then expands the application of the result in [14–16] for the two-dimensional stochastic urban-population growth model. The main result of this paper is illustrated through a numerical simulation.

Notation: The set of (positive) real numbers is denoted by $\mathbb{R}$ ($\mathbb{R}_+$), and the corresponding n-th dimensional (positive orthant) Euclidean space is denoted by $\mathbb{R}^n$ ($\mathbb{R}^n_+$). Given two scalars x_1 and x_2, we define $x_1 \vee x_2 = \max(x_1, x_2)$ and $x_1 \wedge x_2 = \min(x_1, x_2)$. The symbol $|x|$ computes the Euclidean norm of $x \in \mathbb{R}^n$. The symbol $\mathbb{1}_{\{.\}}$ stands for the Dirac measure. Every stochastic process studied in this paper evolves upon a fixed, filtered probability space $(\Omega, \mathcal{F}, P)$.

2. Existence and uniqueness of the stochastic urban-population growth model

This section shows conditions to ensure the existence and uniqueness of solutions for the stochastic urban-population growth model. We emphasize that this paper is not the first to characterize the existence and uniqueness of solutions for such a system [e.g., [14–16]]; however, we show that a condition required by the authors of [14–16] is unnecessary, as detailed in the sequence.

The stochastic model studied in this paper arises from the Itô's extension of the system (1) with $\alpha_i = N_i - d_i/k_i$, $i = 1, 2$, which equals

$$\mathrm{d}x_i(t) = k_i\left(\alpha_i x_i(t) - \beta x_1(t)x_2(t) - x_i^2(t)\right)\mathrm{d}t + \sigma_i x_i(t)\mathrm{d}B_i(t), \quad i = 1, 2, \tag{2}$$

where k_i, α_i, $x_i(0)$, $i = 1, 2$, and β are positive, given scalars. In this paper, the constants $\sigma_i \in \mathbb{R}$, $i = 1, 2$, have no specific sign, in contrast to the studies in [14–16] that require $\sigma_i > 0$, $i = 1, 2$.

Now we recall the meaning of a solution for the system (2).

Definition 2.1 ([17], p. 48; [18], Definition 6.1.3, p. 101). *We say $x_i(t)$, $i = 1, 2$, is a solution for the system (2) if (i) $\{x_i(t)\}$ is continuous and $\mathcal{F}_t$-adapted, and (ii) the equation in (2) is valid for all $t > 0$ with probability one.*

Remark 1. *As proved in the monograph [[18], Coro. 6.3.2, p. 112] any stochastic differential equation with both drift term and diffusion term satisfying the locally Lipschitz condition has a solution within a bounded time frame [see also the proof of Theorem. 2.1 in [15] for a discussion]. As for the stochastic system (1), both the drift terms $x_i \mapsto k_i\alpha_i x_i - k_i\beta x_1 x_2 - k_i x_i^2$, $i = 1, 2$, and the diffusion terms $x_i \mapsto \sigma_i x_i$, $i = 1, 2$, satisfy the local Lipschitz condition. Thus, the result in [[18], Coro. 6.3.2, p. 112] ensures that (2) has a solution $x_i(t)$, $i = 1, 2$.*

Definition 2.2. *We say a solution $x_i(t)$, $i = 1, 2$, is unique if any other solution $\tilde{x}_i(t)$, $i = 1, 2$, is indistinguishable from $x_i(t)$, $i = 1, 2$, that is,*

$$\Pr[x_i(t) = \tilde{x}_i(t) : i = 1, 2, \forall t > 0] = 1.$$

Next, we introduce the concept of *positive solution* for the stochastic population model in (2).

Definition 2.3. *We say the solution $x_i(t)$, $i = 1, 2$, from (2) is positive if, given any initial condition $x_i(0) \in \mathbb{R}_+$, $i = 1, 2$, there holds*

$$\Pr\left[x_i(t) \in \mathbb{R}_+ : i = 1, 2, \forall t > 0\right] = 1.$$

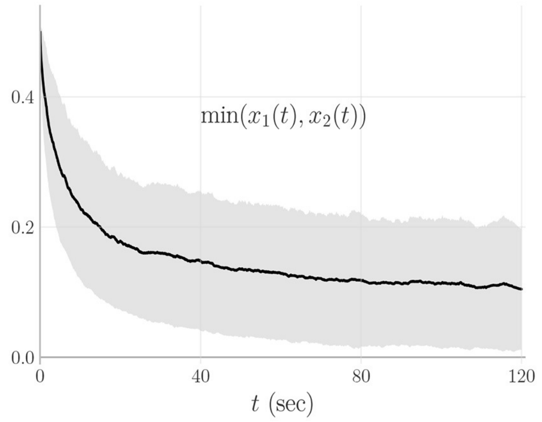

FIGURE 1
Mean (black) and standard deviation (light gray) curves of one thousand sample paths.

Now we can present the main result of this paper.

Theorem 2.1. The solution $x_i(t)$, $i = 1, 2$, from (2) is both positive and unique.

The proof of Theorem 2.1 is postponed to Section 2.2.

Remark 2. *The authors of [14–16] have attained the same result of Theorem 2.1 under the assumption that $\sigma_i > 0$, $i = 1, 2$; however, as stated in Theorem 2.1, this assumption is unnecessary. For this reason, Theorem 2.1 expands the usefulness of the result from [14–16] for the stochastic urban-population growth model as in (2).*

2.1. Numerical simulation

This section illustrates the result of Theorem 2.1 through a simulation. In (1), we set $x_i(0) = 0.5$, $\sigma_i = -1$, $i = 1, 2$, $k_1 = 0.2$, $k_2 = 0.3$, $\alpha_1 = 0.6$, $\alpha_2 = 0.5$, and $\beta = 0.5$. We performed a Monte-Carlo simulation on (1) with one-thousand sample paths, each simulation taking 120 seconds. To simulate (1), we employed the Euler-Maruyama procedure as in [19] with step size of 10^{-5}.

For all the Monte-Carlo samples taken randomly, we observed that both $x_1(t)$ and $x_2(t)$ were positive—this numerical evidence confirms the positiveness of the solution $x_i(t)$, $i = 1, 2$, as discussed in Remark 1. Even though this positiveness is already characterized in the results of [14–16], these results apply only under the condition that $\sigma_i > 0$, $i = 1, 2$. This condition is unnecessary, as discussed in Remark 2; note that the numerical simulation suggested the result hold with $\sigma_i = -1$, $i = 1, 2$.

Figure 1 shows the corresponding mean and standard deviation taken for the minimum value between $x_1(t)$ and $x_2(t)$. The corresponding data indicate that both $x_1(t)$ and $x_2(t)$ are positive and unique, in accordance with Theorem 2.1.

2.2. Proof of Theorem 2.1

Proof. The proof of Theorem 2.1 is divided into two parts. In the first part, we show that $x_i(t)$, $i = 1, 2$, is positive; in the second part, we show that $x_i(t)$, $i = 1, 2$, is unique.

Part I: the solution $x_i(t)$, $i = 1, 2$, from (2) is positive for all $t > 0$.

To see that any solution $x_i(t)$, $i = 1, 2$, satisfying (2) is positive, set $i = 1$ in (2) to write the identity

$$\left(\exp\left(-\int_0^t (k_1\alpha_1 - \beta k_1 x_2(r) - k_1 x_1(r))\mathrm{d}r\right.\right.$$
$$\left.\left.-\sigma_1 \int_0^t \mathrm{d}B_1(r)\right)\right)\frac{\mathrm{d}x_1(t)}{\mathrm{d}t}$$
$$= x_1(t)\left(k_1\alpha_1 - k_1\beta x_2(t) - k_1 x_1(t) + \sigma_1 \frac{\mathrm{d}B_1(t)}{\mathrm{d}t}\right)$$
$$\times \exp\left(-\int_0^t (k_1\alpha_1 - \beta k_1 x_2(r) - k_1 x_1(r))\mathrm{d}r - \sigma_1 \int_0^t \mathrm{d}B_1(r)\right).$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x_1(t)\exp\left(-\int_0^t (k_1\alpha_1 - \beta k_1 x_2(r) - k_1 x_1(r))\mathrm{d}r\right.\right.$$
$$\left.\left.-\sigma_1 \int_0^t \mathrm{d}B_1(r)\right)\right) = 0,$$

which yields

$$x_1(t) = x_1(0)\exp\left(\int_0^t (k_1\alpha_1 - \beta k_1 x_2(r) - k_1 x_1(r))\mathrm{d}r\right.$$
$$\left.-\sigma_1 B_1(t)\right). \tag{3}$$

It then follows from (3) that $x_1(t) \in \mathbb{R}_+$ for all $t > 0$ provided that $x_1(0) \in \mathbb{R}_+$. A similar reasoning applied in (2) with $i = 2$ shows that $x_2(t) \in \mathbb{R}_+$ for all $t > 0$ provided that $x_2(0) \in \mathbb{R}_+$. This argument proves that the stochastic system (2) has a positive solution provided that $x_i(t) \in \mathbb{R}_+$, $i = 1, 2$.

Part II: the solution $x_i(t)$, $i = 1, 2$, from (2) is unique.

To prove the assertion of Part II, we begin with a change of variable upon (2). Namely, set $u_i(t) = \ln x_i(t)$, $i = 1, 2$, and apply upon them the Itô's formula [e.g., [17], p. 36, Theorem 6.4; [20], p. 48, Theorem 4.2.1] to obtain

$$\mathrm{d}u_i(t) = \mathrm{d}\ln x_i(t)$$
$$= \left(k_i\alpha_i - \frac{\sigma_i^2}{2} - \beta k_i \sum_{j=1}^{2} \mathbb{1}_{i\neq j} x_j(t) - k_i x_i(t)\right)\mathrm{d}t$$
$$+\sigma_i \mathrm{d}B_i(t), \tag{4}$$

with $i = 1, 2$. It follows that the solution $u_i(t)$, $i = 1, 2$, from (4) is unique if and only if the solution $x_i(t)$, $i = 1, 2$, from (2) is unique. From now on, we focus our analysis on the solution $u_i(t)$, $i = 1, 2$, from (4).

Define $f_i : \mathbb{R}^2 \mapsto \mathbb{R}$, $i = 1, 2$, as

$$f_i(u_1, u_2) = k_i\alpha_i - \frac{\sigma_i^2}{2} - \beta k_i \sum_{j=1}^{2} \mathbb{1}_{i\neq j} \exp(u_j) - k_i \exp(u_i), \quad i = 1, 2. \tag{5}$$

As a result, the dynamics in (4) is identical to

$$\mathrm{d}u_i(t) = f_i(u_1(t), u_2(t))\mathrm{d}t + \sigma_i \mathrm{d}B_i(t), \quad i = 1, 2, \tag{6}$$

with $u_i(0) = \ln x_i(0)$, $i = 1, 2$.

Now our task is to ensure that the solution $u_i(t)$ has a finite growth whenever the time $t > 0$ is finite. To see that this finite growth holds, let us consider a constant $\eta > 0$ that satisfies $|u| \vee |v| \leq \eta$, where $u = (u_1, u_2) \in \mathbb{R}^2$ and $v = (v_1, v_2) \in \mathbb{R}^2$. As shown in the Appendix, there exists some constant $c = c(\eta) > 0$ such that

$$\max_{i=1,2} |f_i(u_1, u_2) - f_i(v_1, v_2)|^2 \leq c(\eta)|u - v|^2, \tag{7}$$

that is, f_i, $i = 1, 2$, are locally Lipschitz continuous. Let $u_i(t)$, $v_i(t)$, $i = 1, 2$, be two solutions taken from (6). It follows from (6) that

$$u_i(t) - v_i(t) = \int_{t_0}^{t} \big(f_i(u(s)) - f(v(s))\big)\, \mathrm{d}s, \quad \forall t \geq t_0. \tag{8}$$

Applying the expected value operator on both sides of (8), together with the inequality in (7), we obtain (for all $t \in [0, T]$ with given $T > 0$)

$$\mathrm{E}\left[\sup_{t_0 \leq s \leq t} |u(s) - v(s)|^2\right] \leq 2c(\eta)(T + 4)\mathrm{E}\left[\int_0^t \sup_{t_0 \leq r \leq s} |u(r) - v(r)|^2\right] \mathrm{d}s. \tag{9}$$

Finally, the Grönwall's inequality applied in (9) yields [see [17], p. 53]

$$\mathrm{E}\left[\sup_{0 \leq t \leq T} |u(t) - v(t)|^2\right] = 0. \tag{10}$$

The identity in (10) means that $u(t) = v(t)$ for all $t \in [0, T]$, which means that the system (4) has a unique solution on the interval $[0, T]$. As a result, the solution $x_i(t)$, $i = 1, 2$, from (2) is unique when t belongs to the interval $[0, T]$.

It remains to show that $x_i(t)$, $i = 1, 2$, is unique for all $t > T$ when T increases toward infinity. To show this result, we consider the Lyapunov-like function $V : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ as

$$V(x) = x_1 - \ln(x_1) + x_2 - \ln(x_2), \quad \forall x \in \mathbb{R}_+^2. \tag{11}$$

The idea is to use $V(\cdot)$ as in (11) to show that the solution $x_i(t)$, $i = 1, 2$, from (2) cannot diverge to infinity while t is finite.

Using the Itô's formula [e.g., [17], p. 36, Theorem 6.4; [20], p. 48, Theorem 4.2.1] in both (2) and (11) yields

$$\mathrm{d}V(x(t)) = \sum_{i=1}^{2} \left((x_i(t) - 1)\left(k_i\alpha_i - k_i x_i(t) - \beta k_i \sum_{j=1}^{2} \mathbb{1}_{i\neq j} x_j(t) \right) + \frac{\sigma_i^2}{2} \right) \mathrm{d}t + \sigma_i (x_i(t) - 1)\mathrm{d}B_i(t). \tag{12}$$

Since both $x_1(t)$ and $x_2(t)$ are positive (see Part I), we can see that the right-hand side of (12) is bounded from above by

$$\bar{\alpha}(x_1(t) + x_2(t) + 1)\mathrm{d}t + \sum_{i=1}^{2} \sigma_i (x_i(t) - 1)\mathrm{d}B_i(t),$$

where

$$\bar{\alpha} := \max\left\{1, \sum_{i=1}^{2} k_i \left(\alpha_i + 1 + \beta + \frac{\sigma_i^2}{2}\right)\right\}.$$

Taking the expected value operator on both sides of (12), we obtain

$$\mathrm{d}\mathbb{E}[V(x(t))] \leq \bar{\alpha}\left(\mathbb{E}[x_1(t) + x_2(t)] + 1\right) \mathrm{d}t, \quad \forall t \geq 0. \tag{13}$$

Since the expression $x - 2\ln(x)$ is positive when x is positive (see Figure 2), we can write

$$\sum_{i=1}^{2} \mathbb{E}[x_i(t)] \leq \sum_{i=1}^{2} \mathbb{E}[2x_i(t) - 2\ln(x_i(t))]. \tag{14}$$

Substituting (14) into the right-hand side of (13), and considering the definition of $V(\cdot)$ in (12), we can conclude that

$$\mathrm{d}\mathbb{E}[V(x(t))] \leq 2\bar{\alpha}\left(\mathbb{E}[V(x(t))] + 1/2\right) \mathrm{d}t, \quad \forall t \geq 0. \tag{15}$$

Finally, the solution of (15) satisfies

$$\mathbb{E}[V(x(t))] \leq \exp(2\bar{\alpha}t)V(x(0)) + \frac{1}{2\bar{\alpha}}(\exp(2\bar{\alpha}t) - 1), \quad \forall t \geq 0. \tag{16}$$

Even though the term $\mathbb{E}[V(x(t))]$ can increase when t increases, we now know from (16) that the growth of $\mathbb{E}[V(x(t))]$ is limited from above by an exponentially increasing curve. This curve ensures that (2) has a solution $x_i(t)$, $i = 1, 2$, which cannot to diverge to infinity in finite time, as the next argument proves.

To complete the proof, we proceed with a contradiction argument. Suppose from now on that there exists a finite number $T_e > 0$ such that $\max_{i=1,2} x_i(t)$ tends to infinity when t approaches T_e. Let $t_0 > 0$ be the time in which at least either $x_1(t_0)$ or $x_2(t_0)$ is greater than one. Define the stopping times

$$T_n = \inf\left\{t \in [t_0, T_e) : x_i(t) \notin (0, n] \text{ for some } i = 1, 2\right\}, \quad \forall n > 1. \tag{17}$$

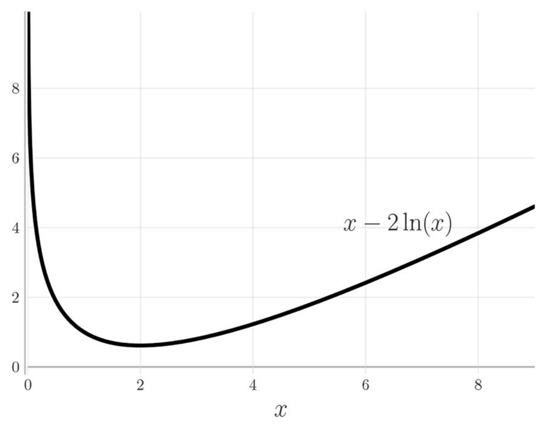

FIGURE 2
The curve indicates that $x - 2\ln(x)$ is positive when x is positive.

It follows from (17) that

$$\max_{i=1,2} x_i(T_n) \to \infty \quad \text{as} \quad n \to \infty, \tag{18}$$

almost surely. Note that each stopping time T_n belongs to the interval $[t_0, T_e)$ and $T_e > 0$ is assumed to be finite. In the next argument, we show that the sequence $\{T_n\}$ diverges to infinity (almost surely) and, as a consequence, that $T_e = \infty$. Note that $T_e = \infty$ contradicts our initial assumption that $T_e > 0$ is finite.

Let us keep our initial assumption that $T_e > 0$ is finite. The fact that $T_n < T_e$ for all $n > 1$ (almost surely) means that

$$\Pr[T_n < T_e] = 1, \quad \forall n > 1. \tag{19}$$

Any realization (i.e., sample-path) of T_n, taken from the underlying sample space Ω, results from (17) that $x(T_n)) > n - \ln(n)$ for each $n > 1$. Thus,

$$V(x(T_n)) > n - \ln(n), \quad \forall n > 1. \tag{20}$$

Combining (16), (19), and (20) yields

$$n - \ln(n) < \mathbb{E}[V(x(T_n))] \le \exp(2\bar{\alpha}T_e)V(x(0)) + \frac{1}{2\bar{\alpha}}\exp(2\bar{\alpha}T_e), \quad \forall n > 1, \tag{21}$$

which is absurd because the term on the left-hand side of (21) tends to infinity when n tends to infinity, while the term on the right-hand side of (21) remains finite. This contradiction proves that $T_e = \infty$, and as a result, the solution $x_i(t)$, $i = 1, 2$, from (2) is unique for all $t > 0$.

3. Concluding remarks

This paper has shown conditions that ensure the positiveness and uniqueness of a stochastic urban-population growth model. This stochastic system has been studied in the literature for n-th dimensional systems [e.g., [14–16]], yet the results available so far require the diffusion constant σ_i be positive. As we have shown in Theorem 2.1, σ_i can be both positive and negative for two-dimensional systems (i.e., $n = 2$). For this reason, Theorem 2.1 can be seen as an extension of the results from [14–16] for two-dimensional systems.

The usefulness of Theorem 2.1 is illustrated through a Monte-Carlo simulation. The simulation was performed for the stochastic urban-population growth model with $\sigma_i = -1$, $i = 1, 2$ (see Section 2.1), and the corresponding data indicate that the system trajectories are both positive and unique—this numerical evidence confirms the novelty of Theorem 2.1.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author/s.

Author contributions

LB, HB, and MD made substantial contributions to the design of the work and generated the data. AV made the interpretation of data and revision of the text. All authors have read and approved the final manuscript.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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Appendix

The function $f_i : \mathbb{R}^2 \mapsto \mathbb{R}$, $i = 1, 2$, defined in (5), reads as

$$f_i(u_1, u_2) = k_i\alpha_i - \frac{k_i^2\sigma_i^2}{2} - \beta k_i \sum_{j=1}^{2} \mathbb{1}_{i \neq j} \exp(u_j) - k_i^2 \exp(u_i),$$
$$i = 1, 2.$$

Hence,

$$\max_{i=1,2} |f_i(u_1, u_2) - f_i(v_1, v_2)|^2 \leq \beta^2 k_1^2 (\exp(v_2) - \exp(u_2))^2$$
$$+ k_1^4 (\exp(v_1) - \exp(u_1))^2 + 2\beta k_1^3 (\exp(v_1) - \exp(u_1))$$
$$(\exp(v_2) - \exp(u_2))$$
$$+ \beta^2 k_2^2 (\exp(v_1) - \exp(u_1))^2 + k_2^4 (\exp(v_2) - \exp(u_2))^2$$
$$+ 2\beta k_2^3 (\exp(v_1) - \exp(u_1))(\exp(v_2) - \exp(u_2)).$$

After a simple algebraic manipulation, we can show that

$$\max_{i=1,2} |f_i(u_1, u_2) - f_i(v_1, v_2)|^2$$
$$\leq \left[\beta(k_1^3 + k_2^3) + (\beta^2 k_2^2 + k_1^4)\right] (\exp(v_1) - \exp(u_1))^2$$
$$+ \left[\beta(k_1^3 + k_2^3) + (\beta^2 k_1^2 + k_2^4)\right] (\exp(v_2) - \exp(u_2))^2. \quad (22)$$

Consider a constant $\eta > 0$ that satisfies $|u| \vee |v| \leq \eta$, where $u = (u_1, u_2) \in \mathbb{R}^2$ and $v = (v_1, v_2) \in \mathbb{R}^2$. According to the Lagrange finite-increments formula, there exists some constant $\xi \in \mathbb{R}$ within the interval from $\min_{i=1,2}(u_i \wedge v_i)$ to $\max_{i=1,2}(u_i \vee v_i)$ such that $|\exp(v_i) - \exp(u_i)| = \exp(\xi)\ |v_i - u_i|$. This fact, together with the inequality in (22), allows us to ensure the existence of some constant $c = c(\eta) > 0$ such that

$$\max_{i=1,2} |f_i(u_1, u_2) - f_i(v_1, v_2)|^2 \leq c(\eta)|u - v|^2. \quad (23)$$

The inequality in (23) means that f_i, $i = 1, 2$, are locally Lipschitz continuous.