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On the Cauchy Problem for Mildly Nonlinear and Non-Boussinesq Case-(ABC) System

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Abstract

In this paper, we investigate the local well-posedness, ill-posedness, and Gevrey regularity of the Cauchy problem for Mildly Nonlinear and Non-Boussinesq case-(ABC) system. The local well-posedness of the solution for this system in Besov spaces $B^{s+1}_{p,r}\times B^{s}_{p,r}$ with $1\le p,r\le\infty$ and $s>\max\left\{1+\frac{1}{p},\frac{3}{2}\right\}$ was firstly established. Next, we consider the continuity of the solution-to-data map, *i.e.* the ill-posedness of the solution for this system in Besov space $B^{s+1}_{p,\infty}\times B^{s}_{p,\infty}$ was derived. Finally, the Gevrey regularity of the system was presented.

Keywords

Local Well-Posedness, Ill-Posedness, Gevrey Regularity

1. Introduction

In this paper, we consider the following Cauchy problem for Mildly Nonlinear and Non-Boussinesq case-(ABC) system (see [1]):

$$\begin{cases}\zeta_t+A\bar{\sigma}_x+\alpha B\left(\zeta\bar{\sigma}\right)_x-\alpha^2C\left(\zeta^2\bar{\sigma}\right)_x=0, & t>0,\ x\in\mathbb{R},\\ \bar{\sigma}_t+\zeta_x+\alpha B\bar{\sigma}\bar{\sigma}_x-\alpha^2C\left(\zeta\bar{\sigma}^2\right)_x=\varepsilon^2\bar{\kappa}\bar{\sigma}_{xxt}, & t>0,\ x\in\mathbb{R},\\ \zeta(x,0)=\zeta_0,\ \bar{\sigma}(x,0)=\bar{\sigma}_0, & t=0,\ x\in\mathbb{R},\end{cases}\tag{1.1}$$

where $\bar{\sigma}$ is the averaged weighted vorticity, and ζ is the interface displacement. α,ε are the small parameters with $\alpha\ll1$ and $\varepsilon^2\ll\alpha\ll\varepsilon$, *i.e.* the Mildly Nonlinear (MNL) case. Moreover, $A,C,\bar{\kappa}$ are the nonnegative parameters, and B is a parameter. For more details, we can refer to [1] [2].

Fluid waves are a ubiquitous phenomenon in marine and atmospheric science. An important cause of fluid internal waves is density stratification. In density-stratified flows, the displacement of a fluid mass from its neutrally buoyant position results in internal wave motion. The dynamics of these internal waves has been of great interest and has been the subject of much research (see e.g. [3] [4]). The study of these long-wave limit currents approaches the existence of a physical environment of rapidly varying density and produces a variety of mathematical models depending on the relative strength of the different influences. The models obtained can be dispersive or non-dispersive and are weakly or fully nonlinear. Physically, dispersion is controlled by the relative magnitude of the horizontal length scale with respect to the height of the domain, however, the nonlinearity is controlled by the wave amplitude with respect to the height of the fluid domain.

Strongly nonlinear, non-dispersive approximations take the form of hyperbolic or mixed-type first-order PDEs, first derived in this context by Long [5]. Weakly nonlinear dispersive approximations result in Korteweg-de Vries type models [6] and fully nonlinear dispersive approximations lead to the so-called Miyata-Camassa-Choi system [7] [8]. However, in this paper, we focus on the Hamiltonian structure of 2-layer dispersed stratified fluids in the non-Boussinesq case under mildly nonlinear assumptions (*i.e.* the Mildly Nonlinear and Non-Boussinesq case-(ABC) system (1.1)).

Moreover, in Ref. [1], solutions of special forms of Equation (1.1), *i.e.* traveling wave solutions and unidirectional waves were considered and the dispersion relation was calculated.

For the equivalent form of (1.1), we set $\bar{\sigma}(x,t)=u(\theta x,t)$ and $\zeta(x,t)=\rho(\theta x,t)$ with $\theta=\frac{1}{\varepsilon\sqrt{\kappa}}$, then we get:

$$\rho_t+\left(\alpha B\theta-2\alpha^2C\theta\rho\right)u\rho_x=-A\theta u_x-\alpha B\theta\rho u_x+\alpha^2C\theta\rho^2u_x,$$

and

$$u_t=\Lambda^{-2}\left(-\theta\rho_x-\alpha B\theta uu_x+\alpha^2C\theta\left(\rho u^2\right)_x\right),$$

where $\Lambda^{2s}u=\left(1-\partial_x^2\right)^s u$ and s is a integer.

In the sequel, we will, for notational convenience, demonstrate local well-posedness of the following initial-value problem with more general coefficients:

$$\begin{cases} u_t=\Lambda^{-2}\left(b_1\rho_x-b_2uu_x+b_3\left(\rho u^2\right)_x\right), & t>0,\ x\in\mathbb{R},\\ \rho_t+\left(b_2u-2b_3\rho u\right)\rho_x=b_4u_x-b_2\rho u_x+b_3\rho^2u_x, & t>0,\ x\in\mathbb{R},\\ u(x,0)=u_0,\ \rho(x,0)=\rho_0, & t=0,\ x\in\mathbb{R},\end{cases} \tag{1.2}$$

with $b_1=-\theta, b_2=\alpha B\theta, b_3=\alpha^2C\theta, b_4=-A\theta$. Where u is the averaged weighted vorticity, and ρ is the interface displacement. Moreover, b_i ($i=1,2,3,4$) are all parameters, the operator Λ is a $\mathcal{S}$-multiplier with $\Lambda^{-2}=\left(1-\partial_x^2\right)^{-1}$.

Inspired by the argument of Danchin [9] [10] in the study of the local well-posedness to the CH equation, we first establish the local well-posedness of (1.2) in Besov spaces.

Theorem 1.1. Suppose that $1 \le p, r \le \infty$ and $s > \max\left\{1+\frac{1}{p}, \frac{3}{2}\right\}$ and the initial data $(u_0, \rho_0) \in B^{s+1}_{p,r} \times B^{s}_{p,r}$. Then, there exists a time $T > 0$ such that the Cauchy problem Equation (2.2) has a unique solution $(u, \rho) \in E^{s+1}_{p,r} \times E^{s}_{p,r}$ (see Definition 2.3). Moreover, the map $(u_0, \rho_0) \mapsto (u, \rho)$ is continuous from a neighborhood of (u_0, ρ_0) in $B^{s+1}_{p,r} \times B^{s}_{p,r}$ into:

$$\mathcal{C}\left([0,T]; B^{s'+1}_{p,r}\right) \cap \mathcal{C}^1\left([0,T]; B^{s'}_{p,r}\right) \times \mathcal{C}\left([0,T]; B^{s'}_{p,r}\right) \cap \mathcal{C}^1\left([0,T]; B^{s'-1}_{p,r}\right)$$

for every $s' < s$ when $r = +\infty$ and $s' = s$ when $r < +\infty$.

Remark 1.2. In the proof of Theorem 1.1, we use the transport equations theory and the transport diffusion equations theory to establish the local well-posedness of Equation (1.2) in nonhomogeneous Besov spaces. It is well-known that the Besov spaces $B^{s+1}_{2,2} \times B^{s}_{2,2}$ coincide with the Sobolev spaces $H^{s+1} \times H^{s}$. Theorem 1.1 implies that under the condition $(u_0, \rho_0) \in H^{s+1} \times H^{s}$ with $s > \frac{3}{2}$, we can obtain the local well-posedness for the data-to-solution map in Sobolev spaces.

In Theorem 1.1, we have proved that for every $s' < s$ when $r = +\infty$ and $s' = s$ when $r < +\infty$, the data-to-solution map of system (1.2) is continue in $B^{s'+1}_{p,r} \times B^{s'}_{p,r}$. Now, another natural question raised: whether or not the data-to-solution map of system (1.2) is continue in $B^{s'+1}_{p,r} \times B^{s'}_{p,r}$ for $s' = s$ and $r = +\infty$. In our next theorem, we shall further show that this data-to-solution map is ill-posedness in $B^{s+1}_{p,\infty} \times B^{s}_{p,\infty}$ in the sense that the solutions starting from (u_0, ρ_0) are discontinuous at $t = 0$ in the metric of $B^{s+1}_{p,\infty} \times B^{s}_{p,\infty}$, this means that:

Theorem 1.3. Suppose that $1 \le p \le \infty$ and $s > \max\left\{1+\frac{1}{p}, \frac{3}{2}\right\}$. Then, the system (1.2) is ill-posed in the Besov spaces $B^{s+1}_{p,\infty} \times B^{s}_{p,\infty}$. More precisely, there exist $(u_0, \rho_0) \in B^{s+1}_{p,\infty} \times B^{s}_{p,\infty}$ and a positive constant δ for which the Cauchy problem of system (1.1) has a unique solution $(u, \rho) \in L^{\infty}\left([0,T]; B^{s+1}_{p,\infty} \times B^{s}_{p,\infty}\right)$ for some $T = T\left(\|u_0\|_{B^{s+1}_{p,r}}, \|\rho_0\|_{B^{s}_{p,r}}\right)$, while

$$\liminf_{t \to 0}\left(\|u - u_0\|_{B^{s+1}_{p,\infty}} + \|\rho - \rho_0\|_{B^{s}_{p,\infty}}\right) \ge C\delta.$$

Motivation by [11] [12], we use the generalized Ovsyannikov theorem to solve the Gevrey and analytic regularity for (1.2). To begin with, we introduce the Sobolev-Gevery spaces [11] [12], a suitable scale of Banach spaces, as follows:

$$G^{\delta}_{\sigma,s} \equiv \left\{ f \in \mathcal{C}^{\infty}(\mathbb{R}) : \|f\|^2_{G^{\delta}_{\sigma,s}} \doteq \int_{\mathbb{R}} \left(1+|\xi|^2\right)^s e^{2\delta|\xi|^{\frac{1}{\sigma}}} \left|\hat{f}(\xi)\right|^2 d\xi < \infty \right\}, \sigma, \delta > 0.$$

It is easy to cheek that $G^{\delta}_{\sigma,s}$ equipped the norm $\|\cdot\|_{G^{\delta}_{\sigma,s}}$ is a Banach space by the completeness of H^s. Then, we can define the Gevrey and analytic regularity as follows:

Definition 1.4. Denoting Fourier multiplier $e^{\delta(-\Delta)^{\frac{1}{2\sigma}}}$ by $e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} f = \mathfrak{I}^{-1}\left(e^{\delta|\xi|^{\frac{1}{\sigma}}}\hat{f}\right)$, *i.e.* $\|f\|_{G^{\delta}_{\sigma,s}} = \left\|e^{\delta|\xi|^{\frac{1}{\sigma}}}\hat{f}\right\|_{H^s}$. If $0<\sigma<1$, it is called ultra-analytic function. If $\sigma=1$, it is usual analytic function and δ is called the radius of analyticity. If $\sigma>1$, it is the Gevrey function.

Now, we establish that solutions of (1.2) are analytic in both space and time variables.

Theorem 1.5. Let $\sigma\ge 1$ and $s>\frac{1}{2}$. Assume that $z_0=(u_0,\rho_0)\in G^1_{\sigma,s}\times G^1_{\sigma,s}$. Therefore, for every $0<\delta<1$, there exists a $T_0>0$ such that the system (1.2) has a unique solution (u,ρ), which is holomorphic in $|t|<\frac{T_0(1-\delta)^{\sigma}}{2^{\sigma}-1}$ with values in $G^{\delta}_{\sigma,s}\times G^{\delta}_{\sigma,s}$.

The rest of our paper is organized as follows. In Section 2, we recall several results in the Littlewood-Paley theory and some properties of Besov spaces are reviewed. In Section 3, we establish the local well-posedness result for Equation (1.2). Moreover, the ill-posedness result for this system is presented in Section 4. Finally, we give the Gevrey regularity in Section 5.

2. Preliminaries

In this section, for the convenience of the readers, we will recall some facts on the Littlewood-Paley theory, which will be frequently used in the following arguments. Then, we introduce some properties of the Besov spaces which will play a key role in proving the local well-posedness and other properties for the system (1.2). One may check [10] [13] for more details. First, we introduce some notations.

Notation. For simplicity, the norm $\|\cdot\|_{B^s_{p,r}}$ means $\|\cdot\|_{B^s_{p,r}(\mathbb{R})}$ in the following, the symbol $A\lesssim B$ means that there is a uniform positive constant C independent of A and B such that $A\le CB$.

Proposition 2.1. (See Proposition 2.10 in [13]) Let $\mathcal{B}\doteq\left\{\xi\in\mathbb{R}^d,|\xi|\le\frac{4}{3}\right\}$ and $\mathcal{C}\doteq\left\{\xi\in\mathbb{R}^d,\frac{4}{3}\le|\xi|\le\frac{8}{3}\right\}$. There exist two radial functions $\chi\in C_c^{\infty}(\mathcal{B})$ and $\varphi\in C_c^{\infty}(\mathcal{C})$ such that:

$$\chi(\xi)+\sum_{q\ge 0}\varphi\left(2^{-q}\xi\right)=1,\ \text{for all } \xi\in\mathbb{R}^d,$$

$$|q-q'|\ge 2\Rightarrow Supp\varphi\left(2^{-q}\cdot\right)\cap Supp\varphi\left(2^{-q'}\cdot\right)=\varnothing,$$

$$q\ge 1\Rightarrow Supp\chi(\cdot)\cap Supp\varphi\left(2^{-q'}\cdot\right)=\varnothing,$$

$$\frac{1}{3}\le\chi(\xi)^2+\sum_{q\ge 0}\varphi\left(2^{-q}\xi\right)^2\le 1,\ \text{for all } \xi\in\mathbb{R}^d.$$

Moreover, let $h\doteq\mathcal{F}^{-1}\varphi$ and $\tilde{h}\doteq\mathcal{F}^{-1}\chi$. Then, for all $f\in\mathcal{S}'\left(\mathbb{R}^d\right)$, the dyadic operators Δ_q and S_q can be defined as follows:

$$\Delta_q f \doteq \varphi\left(2^{-q}D\right) f = 2^{qd}\int_{\mathbb{R}^d} h\left(2^q y\right) f\left(x-y\right)\mathrm{d}y,\ \text{for } q\geq 0,$$

$$S_q f \doteq \chi\left(2^{-q}D\right) f = \sum_{-1\leq k\leq q-1}\Delta_k = 2^{qd}\int_{\mathbb{R}^d}\tilde{h}\left(2^q y\right) f\left(x-y\right)\mathrm{d}y,$$

$$\Delta_{-1} f \doteq S_0 f \text{ and } \Delta_q f \doteq 0 \text{ for } q\leq -2.$$

Therefore,

$$f = \sum_{q\geq 0}\Delta_q f \text{ in } S'\left(\mathbb{R}^d\right),$$

and the right-hand side is called the nonhomogeneous Littlewood-Paley decomposition of *f*.

Definition 2.2. (See Definition 2.68 in [13]) Let $s\in\mathbb{R}$, $1\leq p,r\leq\infty$. The inhomogenous Besov space $B^s_{p,r}\left(\mathbb{R}^d\right)$ ($B^s_{p,r}$ for short) is defined by:

$$B^s_{p,r} \doteq \left\{ f\in\mathcal{S}'\left(\mathbb{R}^d\right); \|f\|_{B^s_{p,r}} < \infty\right\},$$

where

$$\|f\|_{B^s_{p,r}} \doteq \begin{cases}\left(\sum_{q\in\mathbb{Z}} 2^{qsr}\left\|\Delta_q f\right\|^r_{L_p}\right)^{\frac{1}{r}}, & \text{for } r<\infty,\\ \sup_{q\in\mathbb{Z}} 2^{qs}\left\|\Delta_q f\right\|_{L_p}, & \text{for } r=\infty.\end{cases}$$

If $s=\infty$, $B^\infty_{p,r} \doteq \bigcap_{s\in\mathbb{R}} B^s_{p,r}$.

Definition 2.3. For $T>0, s\in\mathbb{R}$ and $1\leq p\leq +\infty$, we define:

$$E^s_{p,r}\left(T\right) \doteq C\left(\left[0,T\right]; B^s_{p,r}\right)\cap C^1\left(\left[0,T\right]; B^{s-1}_{p,r}\right),\ \text{if } r<\infty,$$

$$E^s_{p,\infty}\left(T\right) \doteq L^\infty\left(\left[0,T\right]; B^s_{p,\infty}\right)\cap Lip\left(\left[0,T\right]; B^{s-1}_{p,\infty}\right) \text{ and } E^s_{p,r} \doteq \bigcap_{T>0} E^s_{p,r}\left(T\right).$$

Proposition 2.4. (See Corollary 2.86 in [13]) For any positive real number s and any $\left(p,r\right)$ in $\left[1,\infty\right]^2$, the space $L^\infty\left(\mathbb{R}^d\right)\cap B^s_{p,r}\left(\mathbb{R}^d\right)$ is an algebra and a constant C exists such that:

$$\|uv\|_{B^s_{p,r}\left(\mathbb{R}^d\right)} \leq C\left(\|u\|_{L^\infty\left(\mathbb{R}^d\right)}\|v\|_{B^s_{p,r}\left(\mathbb{R}^d\right)} + \|u\|_{B^s_{p,r}\left(\mathbb{R}^d\right)}\|v\|_{L^\infty\left(\mathbb{R}^d\right)}\right).$$

If $s>\frac{d}{p}$ or $s=\frac{d}{p}$, $r=1$, then we have:

$$\|uv\|_{B^s_{p,r}\left(\mathbb{R}^d\right)} \leq C\|u\|_{B^s_{p,r}\left(\mathbb{R}^d\right)}\|v\|_{B^s_{p,r}\left(\mathbb{R}^d\right)}.$$

Proposition 2.5. (See Proposition 1.3.5 in [10]) Suppose that $s\in\mathbb{R}$, $1\leq p,r,p_i,r_i\leq\infty$ ($i=1,2$). We have:

1) Topological properties: $B^s_{p,r}$ is a Banach space and is continuously embedded in S'.

2) Density: C^∞_c is dense in $B^s_{p,r} \Leftrightarrow 1\leq p,r<\infty$.

3) Embedding: $B^s_{p_1,r_1} \hookrightarrow B^{s-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}$, if $p_1\leq p_2$ and $r_1\leq r_2$. $B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$ locally compact, if $s_1<s_2$.

4) Algebraic properties: for all $s>0$, $B^s_{p,r}\cap L^\infty$ is an algebra. Moreover, $B^s_{p,r}$ is an algebra $\Leftrightarrow B^s_{p,r}\hookrightarrow L^\infty \Leftrightarrow s>\frac{n}{p}$ (or $s=\frac{n}{p}$ and $r=1$).

5) Complex interpolation:

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}}\le C\|u\|^{\theta}_{B^{s_1}_{p,r}}\|u\|^{1-\theta}_{B^{s_2}_{p,r}},$$

for all $u\in B^{s_1}_{p,r}\cap B^{s_2}_{p,r}$, for all $\theta\in[0,1]$.

6) Fatou lemma: If $(u_n)_{n\in\mathbb{N}}$ is bounded in $B^s_{p,r}$ and $u_n\to u$ in S', then $u\in B^s_{p,r}$ and

$$\|u\|_{B^s_{p,r}}\le\liminf_{n\to\infty}\|u_n\|_{B^s_{p,r}}.$$

7) Let $m\in\mathbb{R}$ and let f be an S^m-multiplier (*i.e.* $f:\mathbb{R}^d\to\mathbb{R}$ is smooth and satisfies that for all $\alpha\in\mathbb{N}^d$, there exists a constant C_α, s.t. $\left|\partial^\alpha f(\xi)\right|\le C_\alpha\left(1+|\xi|^{m-|\alpha|}\right)$ for all $\xi\in\mathbb{R}^d$). Then, the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.

Lemma 2.6. (See Lemma 2.8 in [14] or [13]) Suppose that $(p,r)\in[1,+\infty]^2$ and $s>-\min\left\{\frac{1}{p},1-\frac{1}{p}\right\}$. Assume $f_0\in B^s_{p,r}$, $F\in L^1([0,T];B^s_{p,r})$, and

$$\begin{cases}\partial_x v\in L^1([0,T];B^{s-1}_{p,r}); & \text{if } s>1+\frac{1}{p}\left(s=1+\frac{1}{p},r=1\right),\\ \partial_x v\in L^1\left([0,T];B^{\frac{1}{p}}_{p,r}\cap L^\infty\right); & \text{if } s<1+\frac{1}{p}.\end{cases}$$

If $f\in L^\infty([0,T];B^s_{p,r})\cap\mathcal{C}([0,T];\mathcal{S}')$ solves the following 1-D linear transport equation:

$$\begin{cases}\partial_t f+v\cdot\partial_x f=F,\\ f|_{t=0}=f_0,\end{cases}\tag{2.1}$$

then exists a constant C depending only on p,r,s, such that the following statements hold:

1) If $r=1$ or $s\ne1+\frac{1}{p}$,

$$\|f(t)\|_{B^s_{p,r}}\le\|f_0\|_{B^s_{p,r}}+\int_0^t\|F(\tau)\|_{B^s_{p,r}}+CV'(\tau)\|f(\tau)\|_{B^s_{p,r}}\,\mathrm{d}\tau,$$

or

$$\|f(t)\|_{B^s_{p,r}}\le\mathrm{e}^{CV(t)}\left(\|f_0\|_{B^s_{p,r}}+\int_0^t\mathrm{e}^{-CV(\tau)}\|F(\tau)\|_{B^s_{p,r}}\,\mathrm{d}\tau\right),$$

with

$$V(t)=\begin{cases}\int_0^t\|\partial_x v\|_{B^{\frac{1}{p}}_{p,r}\cap L^\infty}\,\mathrm{d}\tau, & \text{if } s<1+\frac{1}{p};\\ \int_0^t\|\partial_x v\|_{B^{s-1}_{p,r}}\,\mathrm{d}\tau, & \text{if } s>1+\frac{1}{p}\left(\text{or } s=1+\frac{1}{p},r=1\right).\end{cases}$$

2) If $f=v$, then for all $s>0$, 1) holds true with $V(t)=\int_0^t \|\partial_x v\|_{L^\infty}\,\mathrm{d}\tau$.

3) If $r<\infty$, then $f\in\mathcal{C}([0,T];B^s_{p,r})$. If $r=\infty$, then $f\in\mathcal{C}([0,T];B^{s'}_{p,1})$ for all $s'<s$.

Lemma 2.7. (See Theorem 3.3.1 in [10] or [15]) Let $(p,p_1,r)\in[0,T]^3$, and let $s>-\min\left(\frac{1}{p_1},\frac{1}{p'}\right)$ with $p'\doteq\left(1-\frac{1}{p}\right)^{-1}$. Assume that $f_0\in B^s_{p,r}$, $F\in L^1([0,T];B^s_{p,r})$. Let v be a time-dependent vector field such that $v\in L^\rho([0,T];B^{-M}_{\infty,\infty})$ for some $\rho>1$, $M>0$ and

$$\partial_x v\in L^1\left([0,T];B^{\frac{1}{p_1}}_{p_1,\infty}\right),\ \text{if } s<1+\frac{1}{p_1},$$

and

$$\partial_x v\in L^1\left([0,T];B^{s-1}_{p_1,r}\right),\ \text{if } s>1+\frac{1}{p_1},\text{ or}\left(s=1+\frac{1}{p_1},r=1\right).$$

Then, Equation (2.1) has a unique solution $f\in L^\infty([0,T];B^s_{p,r})\cap\left(\bigcap_{s'<s}\mathcal{C}([0,T];B^{s'}_{p,1})\right)$ and the inequalities in Lemma 2.6 hold true.

Moreover, if $r<\infty$, then we have $f\in\mathcal{C}([0,T];B^s_{p,r})$.

Lemma 2.8. (See [13]) Let $s>0$, $1\le p\le\infty$, then we have:

$$\left\|2^{js}\left\|[\Delta_j,u]\partial_x v\right\|_{L^p}\right\|_{l^\infty}\le C\left(\|\partial_x u\|_{L^\infty}\|v\|_{B^s_{p,\infty}}+\|\partial_x v\|_{L^\infty}\|u\|_{B^s_{p,\infty}}\right),$$

where $[\Delta_j,u]\partial_x v=\Delta_j(u\partial_x v)-u\Delta_j(\partial_x v)$.

3. Local Well-Posedness in Besov Space

In this section, we shall discuss the local well-posedness of the Cauchy problem (1.2) in nonhomogeneous Besov spaces, and prove Theorem 1.1. In the following, we denote $C>0$ a generic constant only depending on p,r,s,b_1,b_2,b_3,b_4. Moreover, uniqueness and continuity with respect to the initial data (u_0,ρ_0) are an immediate consequence of the following lemma.

Lemma 3.1. Let $1\le p,r\le+\infty$ and $s>\max\left\{1+\frac{1}{p},\frac{3}{2}\right\}$. Suppose that:

$$(u_i,\rho_i)\in\left\{L^\infty([0,T];B^{s+1}_{p,r})\cap\mathcal{C}([0,T];\mathcal{S}')\right\}\times\left\{L^\infty([0,T];B^s_{p,r})\cap\mathcal{C}([0,T];\mathcal{S}')\right\}(i=1,2)$$

be two given solutions of the initial-value problem (1.2) with the initial data $(u_i(0),\rho_i(0))\in B^{s+1}_{p,r}\times B^s_{p,r}(i=1,2)$, and denote $u_{12}=u_1-u_2$, $\rho_{12}=\rho_1-\rho_2$. Then, for every $t\in[0,T]$, we have:

$$\|u_{12}\|_{B^s_{p,r}}+\|\rho_{12}\|_{B^{s-1}_{p,r}}\le\left(\|u_{12}(0)\|_{B^s_{p,r}}+\|\rho_{12}(0)\|_{B^{s-1}_{p,r}}\right)\exp\left(C\int_0^t\Gamma_s(\tau)\,\mathrm{d}\tau\right),\quad(3.1)$$

for $s\neq 2+\frac{1}{p}$, where

$$\Gamma_s(t)=\left(1+\|u_1\|_{B^{s+1}_{p,r}}+\|u_2\|_{B^{s+1}_{p,r}}+\|\rho_1\|_{B^{s}_{p,r}}+\|\rho_2\|_{B^{s}_{p,r}}\right)^2.$$

For the critical case $s=2+\frac{1}{p}$,

$$\|u_{12}\|_{B^{2+\frac{1}{p}}_{p,q}}+\|\rho_{12}\|_{B^{1+\frac{1}{p}}_{p,q}}\le C\left(\|u_{12}(0)\|_{B^{2+\frac{1}{p}}_{p,q}}+\|\rho_{12}(0)\|_{B^{1+\frac{1}{p}}_{p,q}}\right)^{\theta}\exp\left(C\theta\int_0^T\Gamma_{3+\frac{1}{p}}\,\mathrm{d}\tau\right)\Gamma^{\frac{1-\theta}{2}}_{3+\frac{1}{p}},$$

where $\theta=1-\frac{1}{2p}\in(0,1)$.

Proof (Proof of Lemma 3.1) It is easy to see that $u_{12}\in L^\infty\left([0,T];B^{s+1}_{p,r}\right)\cap\mathcal{C}\left([0,T];\mathcal{S}'\right)$ and $\rho_{12}\in L^\infty\left([0,T];B^{s}_{p,r}\right)\cap\mathcal{C}\left([0,T];\mathcal{S}'\right)$ which implies that the solution pair $(u_{12},\rho_{12})\in\mathcal{C}\left([0,T];B^{s}_{p,r}\right)\times\mathcal{C}\left([0,T];B^{s-1}_{p,r}\right)$, and (u_{12},ρ_{12}) solves the following transport equations:

$$\begin{cases}\partial_t u_{12}=\Lambda^{-2}\partial_x H(u_1,u_2,\rho_1,\rho_2),\\ \partial_t\rho_{12}+(b_2u_1-2b_3\rho_1u_1)\partial_x\rho_{12}=G(u_1,u_2,\rho_1,\rho_2),\\ u_{12}|_{t=0}=u_1(0)-u_2(0),\ \rho_{12}|_{t=0}=\rho_1(0)-\rho_2(0),\end{cases}\tag{3.2}$$

where

$$H=b_1\rho_{12}-\frac{b_2}{2}(u_1+u_2)u_{12}+b_3\left(\rho_{12}u_1^2+\rho_2(u_1+u_2)u_{12}\right),$$

and

$$\begin{aligned}G=&-b_2u_{12}\partial_x\rho_2+2b_3\partial_x\rho_2(\rho_1u_{12}+u_2\rho_{12})+b_4\partial_xu_{12}\\&-b_2(\rho_1\partial_xu_{12}+\rho_{12}\partial_xu_2)+b_3\left[\rho_1^2\partial_xu_{12}+\partial_xu_2\rho_{12}(\rho_1+\rho_2)\right].\end{aligned}$$

Integrating the first equation of (3.2) with respect to variable *t*, it is easy to see that:

$$\|u_{12}\|_{B^s_{p,r}}\le\|u_{12}(0)\|_{B^s_{p,r}}+\int_0^t\|\Lambda^{-2}\partial_xH\|_{B^s_{p,r}}\,\mathrm{d}\tau.$$

Note that the operator Λ is a $\mathcal{S}$-multiplier. Applying Proposition 2.4, Proposition 2.5 and the algebraic property for $B^{s-1}_{p,r}$ for $s>1+\frac{1}{p}$, we have:

$$\begin{aligned}\|\Lambda^{-2}\partial_xH\|_{B^s_{p,r}}&\le C\left\|b_1\rho_{12}-\frac{b_2}{2}(u_1+u_2)u_{12}+b_3\left(\rho_{12}u_1^2+\rho_2(u_1+u_2)u_{12}\right)\right\|_{B^{s-1}_{p,r}}\\&\le C\left(\|\rho_{12}\|_{B^{s-1}_{p,r}}+\|(u_1+u_2)u_{12}\|_{B^{s-1}_{p,r}}+\|\rho_{12}u_1^2\|_{B^{s-1}_{p,r}}+\|\rho_2(u_1+u_2)u_{12}\|_{B^{s-1}_{p,r}}\right)\\&\le C\left(\left(\|u_{12}\|_{B^s_{p,r}}+\|\rho_{12}\|_{B^{s-1}_{p,r}}\right)\Gamma_s(\tau)\right)\end{aligned}$$

Therefore, we obtain:

$$\|u_{12}\|_{B^s_{p,r}}\le\|u_{12}(0)\|_{B^s_{p,r}}+C\int_0^t\left(\|u_{12}\|_{B^s_{p,r}}+\|\rho_{12}\|_{B^{s-1}_{p,r}}\right)\Gamma_s(\tau)\,\mathrm{d}\tau.\tag{3.3}$$

Applying the Lemma 2.6 for $s>\max\left\{1+\frac{1}{p},\frac{3}{2}\right\}$ and $s\ne2+\frac{1}{p}$, we get the following inequality:

$$\|\rho_{12}\|_{B^{s-1}_{p,r}} \le \|\rho_{12}(0)\|_{B^{s-1}_{p,r}} + \int_0^t \|G(\tau)\|_{B^{s-1}_{p,r}} + CV'(\tau)\|\rho_{12}\|_{B^{s-1}_{p,r}} \mathrm{d}\tau, \tag{3.4}$$

with

$$V(t) = \begin{cases} \int_0^t \left\|\partial_x (b_2 u_1 - 2b_3\rho_1 u_1)\right\|_{B^{\frac{1}{p}}_{p,r}\cap L^\infty} \mathrm{d}\tau, & \text{if } \max\left\{1+\frac{1}{p},\frac{3}{2}\right\} < s < \frac{5}{2}, \\ \int_0^t \left\|\partial_x (b_2 u_1 - 2b_3\rho_1 u_1)\right\|_{B^{s-2}_{p,r}} \mathrm{d}\tau, & \text{if } s \ge \frac{5}{2}. \end{cases}$$

Using Proposition 2.5 (3) ($B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$ locally compact, if $s_1 < s_2$) and Proposition 2.5 (4) ($B^{s}_{p,r} \hookrightarrow L^\infty \Leftrightarrow s > \frac{n}{p}$), for $s > \max\left\{1+\frac{1}{p},\frac{3}{2}\right\}$, we get:

$$\begin{aligned} V'(t) &\le \left\|\partial_x (b_2 u_1 - 2b_3\rho_1 u_1)\right\|_{B^{\frac{1}{p}}_{p,r}\cap L^\infty} + \left\|\partial_x (b_2 u_1 - 2b_3\rho_1 u_1)\right\|_{B^{s-2}_{p,r}} \\ &\le 2\left\|\partial_x (b_2 u_1 - 2b_3\rho_1 u_1)\right\|_{B^{s-1}_{p,r}} \\ &\le 2C\left(\|\rho_1\|_{B^{s}_{p,r}}\|u_1\|_{B^{s+1}_{p,r}} + \|u_1\|_{B^{s+1}_{p,r}}\right) \end{aligned} \tag{3.5}$$

For $s > \max\left\{1+\frac{1}{p},\frac{3}{2}\right\}$ and $s \ne 2+\frac{1}{p}$, applying Proposition 2.4, we can obtain that:

$$\begin{aligned} \|G\|_{B^{s-1}_{p,r}} &\le C\Big(\|u_{12}\|_{B^{s}_{p,r}}\|\rho_2\|_{B^{s}_{p,r}} + \|u_{12}\|_{B^{s}_{p,r}} + \|\rho_2\|_{B^{s}_{p,r}}\Big(\|\rho_1\|_{B^{s}_{p,r}}\|u_{12}\|_{B^{s}_{p,r}} \\ &\quad + \|\rho_{12}\|_{B^{s-1}_{p,r}}\|u_2\|_{B^{s+1}_{p,r}}\Big) + \|u_{12}\|_{B^{s}_{p,r}}\|\rho_1\|_{B^{s}_{p,r}} + \|\rho_{12}\|_{B^{s-1}_{p,r}}\|u_2\|_{B^{s+1}_{p,r}} \\ &\quad + \|u_{12}\|_{B^{s}_{p,r}}\|\rho_1\|^2_{B^{s}_{p,r}} + \|u_2\|_{B^{s+1}_{p,r}}\|\rho_{12}\|_{B^{s-1}_{p,r}}\Big(\|\rho_1\|_{B^{s}_{p,r}} + \|\rho_2\|_{B^{s}_{p,r}}\Big)\Big) \\ &\le C\Big(\Big(\|u_{12}\|_{B^{s}_{p,r}} + \|\rho_{12}\|_{B^{s-1}_{p,r}}\Big)\Gamma_s(\tau)\Big). \end{aligned} \tag{3.6}$$

Submitting (3.5)-(3.6) into (3.4), we may derive:

$$\begin{aligned} \|\rho_{12}\|_{B^{s-1}_{p,r}} &\le \|\rho_{12}(0)\|_{B^{s-1}_{p,r}} + \int_0^t \|G(\tau)\|_{B^{s-1}_{p,r}} + CV'(\tau)\|\rho_{12}\|_{B^{s-1}_{p,r}} \mathrm{d}\tau \\ &\le \|\rho_{12}(0)\|_{B^{s-1}_{p,r}} + C\int_0^t \Big(\|u_{12}\|_{B^{s}_{p,r}} + \|\rho_{12}\|_{B^{s-1}_{p,r}}\Big)\Gamma_s(\tau)\mathrm{d}\tau \end{aligned} \tag{3.7}$$

for $s > \max\left\{1+\frac{1}{p},\frac{3}{2}\right\}$ and $s \ne 2+\frac{1}{p}$.

Consequently, combining (3.3) and (3.7), we have:

$$\begin{aligned} &\|u_{12}\|_{B^{s}_{p,r}} + \|\rho_{12}\|_{B^{s-1}_{p,r}} \\ &\le \|u_{12}(0)\|_{B^{s}_{p,r}} + \|\rho_{12}(0)\|_{B^{s-1}_{p,r}} + C\int_0^t \Big(\|u_{12}\|_{B^{s}_{p,r}} + \|\rho_{12}\|_{B^{s-1}_{p,r}}\Big)\Gamma_s(\tau)\mathrm{d}\tau. \end{aligned}$$

Applying Gronwall's lemma to the above inequality leads to (3.1).

For the case $s = 2+\frac{1}{p}$, choosing $\theta = 1-\frac{1}{2p} \in (0,1)$, then we have:

$$1+\frac{1}{p}=\theta\left(1+\frac{1}{2p}\right)+(1-\theta)\left(2+\frac{1}{2p}\right),$$

and

$$2+\frac{1}{p}=\theta\left(2+\frac{1}{2p}\right)+(1-\theta)\left(3+\frac{1}{2p}\right).$$

And according to the interpolation formula in Proposition 2.5 (5) and the obtained result (3.1), we can see that:

$$\begin{aligned}
&\|u_{12}\|_{B_{p,q}^{2+\frac{1}{p}}}+\|\rho_{12}\|_{B_{p,q}^{1+\frac{1}{p}}}\le\|u_{12}\|^{\theta}_{B_{p,q}^{2+\frac{1}{2p}}}\|u_{12}\|^{1-\theta}_{B_{p,q}^{3+\frac{1}{2p}}}+\|\rho_{12}\|^{\theta}_{B_{p,q}^{1+\frac{1}{2p}}}\|\rho_{12}\|^{1-\theta}_{B_{p,q}^{2+\frac{1}{2p}}}\\
&\le\left(\|u_{12}\|_{B_{p,q}^{2+\frac{1}{2p}}}+\|\rho_{12}\|_{B_{p,q}^{1+\frac{1}{2p}}}\right)^{\theta}\left(\|u_{12}\|^{1-\theta}_{B_{p,q}^{3+\frac{1}{2p}}}+\|\rho_{12}\|^{1-\theta}_{B_{p,q}^{2+\frac{1}{2p}}}\right)\\
&\le\left(\|u_{12}\|_{B_{p,q}^{2+\frac{1}{2p}}}+\|\rho_{12}\|_{B_{p,q}^{1+\frac{1}{2p}}}\right)^{\theta}\left(\|u_{1}\|_{B_{p,q}^{3+\frac{1}{2p}}}+\|u_{2}\|_{B_{p,q}^{3+\frac{1}{2p}}}+\|\rho_{1}\|_{B_{p,q}^{2+\frac{1}{2p}}}+\|\rho_{2}\|_{B_{p,q}^{2+\frac{1}{2p}}}\right)^{1-\theta}\\
&\le C\left(\|u_{12}(0)\|_{B_{p,q}^{2+\frac{1}{p}}}+\|\rho_{12}(0)\|_{B_{p,q}^{1+\frac{1}{p}}}\right)^{\theta}\exp\left(C\theta\int_0^T\Gamma_{3+\frac{1}{p}}\mathrm{d}\tau\right)\Gamma^{\frac{1-\theta}{2}}_{3+\frac{1}{p}}
\end{aligned}$$

which yields the Lemma 3.1.

Now, let us start the proof of Theorem 1.1, which is motivated by the proof of local existence theorem about Camassa-Holm type equations in [15] [16]. Next, by using the classical Friedrichs regularization method, we construct the approximate solutions to (1.2).

Lemma 3.2. Let p, r and s be as in the statement of Lemma 2.6. Assume that $u(0)=\rho(0):=0$. There exists a sequence of smooth functions $(u_k,\rho_k)\in\mathcal{C}\left(\mathbb{R}^+;B^{\infty}_{p,r}\right)\times\mathcal{C}\left(\mathbb{R}^+;B^{\infty}_{p,r}\right)$ solving:

$$\begin{cases}\partial_t u_{k+1}=\Lambda^{-2}\partial_x H_k(u_k,\rho_k),\\ \partial_t\rho_{k+1}+(b_2u_k-2b_3\rho_ku_k)\partial_x\rho_{k+1}=G_k(u_k,\rho_k),\\ u_{k+1}(0)=S_{k+1}u(0),\rho_{k+1}(0)=S_{k+1}\rho(0),\end{cases}\tag{3.8}$$

where

$$H_k=b_1\rho_k-\frac{b_2}{2}u_k^2+b_3\rho_ku_k^2,$$

and

$$G_k=b_4\partial_xu_k-b_2\rho_k\partial_xu_k+b_3\rho_k^2\partial_xu_k.$$

Moreover, there is a positive time T such that the solutions satisfy the following properties:

1) $(u_k,\rho_k)_{k\in\mathbb{N}}$ is uniformly bounded in $E^{s+1}_{p,r}(T)\times E^{s}_{p,r}(T)$.
2) $(u_k,\rho_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}\left([0,T];B^{s}_{p,r}\right)\times\mathcal{C}\left([0,T];B^{s-1}_{p,r}\right)$.

Proof (Proof of Lemma 3.2) Since all the data $S_{k+1}u_0$ and $S_{k+1}\rho_0$ belong to $B^{\infty}_{p,r}$, Lemma 2.7 indicates that for all $k\in\mathbb{N}$, Equation (3.8) has a global solution in $\mathcal{C}\left(\mathbb{R}^+;B^{\infty}_{p,r}\right)\times\mathcal{C}\left(\mathbb{R}^+;B^{\infty}_{p,r}\right)$.

Obviously, we can get the following inequality:

$$\begin{aligned}\|u_{k+1}\|_{B^{s+1}_{p,r}} &\le \|u(0)\|_{B^{s+1}_{p,r}} + \int_0^t \left\|\Lambda^{-2}\partial_x H_k\right\|_{B^{s+1}_{p,r}} \mathrm{d}\tau \\ &\le \|u(0)\|_{B^{s+1}_{p,r}} + C\int_0^t \|\rho_k\|_{B^s_{p,r}} + \|u_k\|^2_{B^{s+1}_{p,r}} + \|\rho_k\|_{B^s_{p,r}}\|u_k\|^2_{B^{s+1}_{p,r}} \mathrm{d}\tau.\end{aligned}$$

Using Lemma 2.6 again, for $s > \max\left\{1+\frac{1}{p},\frac{3}{2}\right\}$, we can see that:

$$\|\rho_{k+1}\|_{B^s_{p,r}} \le e^{CV(t)}\left(\|\rho(0)\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|G_k(\tau)\|_{B^s_{p,r}} \mathrm{d}\tau\right),$$

with

$$\begin{aligned}V(t) &= \int_0^t \left\|\partial_x\left(b_2u_k - 2b_3\rho_k u_k\right)\right\|_{B^{s-1}_{p,r}} \mathrm{d}\tau \\ &\le C\int_0^t \|u_k\|_{B^{s+1}_{p,r}} + \|\rho_k\|_{B^s_{p,r}}\|u_k\|_{B^{s+1}_{p,r}} \mathrm{d}\tau.\end{aligned}$$

And with the help of Proposition 2.4 again, we have:

$$\|G_k(\tau)\|_{B^s_{p,r}} \le C\left(\|u_k\|_{B^{s+1}_{p,r}} + \|\rho_k\|_{B^s_{p,r}}\|u_k\|_{B^{s+1}_{p,r}} + \|\rho_k\|^2_{B^s_{p,r}}\|u_k\|_{B^{s+1}_{p,r}}\right).$$

Combining the above inequalities, we get:

$$\begin{aligned}\|\rho_{k+1}\|_{B^s_{p,r}} &\le \exp\left[C\int_0^t \|u_k\|_{B^{s+1}_{p,r}}\left(1+\|\rho_k\|_{B^s_{p,r}}\right)\mathrm{d}\tau\right]\|\rho(0)\|_{B^s_{p,r}} \\ &\quad + \int_0^t \exp\left[C\int_\tau^t \|u_k\|_{B^{s+1}_{p,r}}\left(1+\|\rho_k\|_{B^s_{p,r}}\right)\mathrm{d}\tau'\right] \\ &\quad \left[\|u_k\|_{B^{s+1}_{p,r}}\left(1+\|\rho_k\|_{B^s_{p,r}} + \|\rho_k\|^2_{B^s_{p,r}}\right)\right]\mathrm{d}\tau,\end{aligned}$$

Therefore, if we define $U_k(t) = \|u_k\|_{B^{s+1}_{p,r}} + \|\rho_k\|_{B^s_{p,r}} + 1$ and $U_0 = \|u(0)\|_{B^{s+1}_{p,r}} + \|\rho(0)\|_{B^s_{p,r}} + 1$, then

$$\|u_{k+1}\|_{B^{s+1}_{p,r}} \le U_0 + C\int_0^t U_k^3 d\tau, \tag{3.9}$$

and

$$\|\rho_{k+1}\|_{B^s_{p,r}} \le \exp\left(C\int_0^t U_k^2(\tau)\mathrm{d}\tau\right)U_0 + C\int_0^t \exp\left(C\int_\tau^t U_k^2(\tau')\mathrm{d}\tau'\right)U_k^3\mathrm{d}\tau. \tag{3.10}$$

Choosing $T^* = \frac{3}{256CU_0^2}$, by induction, we show that:

$$U_k \le \frac{4U_0}{\sqrt{1-64CU_0^2 t}}, \forall t \in \left[0, T^*\right). \tag{3.11}$$

In fact, suppose that (3.11) is valid for k, then for $0 \le \tau < t < T^* = \frac{3}{256CU_0^2} < \frac{1}{64CU_0^2}$, we have:

$$\exp\left(C\int_\tau^t U_k^2(\tau')\mathrm{d}\tau'\right) \le \exp\left(\int_\tau^t \frac{16CU_0^2}{1-64CU_0^2\tau'}\mathrm{d}\tau'\right) = \left(\frac{1-64CU_0^2\tau}{1-64CU_0^2 t}\right)^{\frac{1}{4}}, \tag{3.12}$$

and it is easy to see that:

$$1 \le \frac{1}{\left(1-64CU_0^2 t\right)^{\frac{1}{4}}}.$$

Submitting (3.11) and (3.12) into (3.9) and (3.10), we obtain:

$$\begin{aligned}
\|u_{k+1}\|_{B^{s+1}_{p,r}} + \|\rho_{k+1}\|_{B^s_{p,r}} &\le \frac{1}{\left(1-64CU_0^2 t\right)^{\frac{1}{4}}} \|u_{k+1}\|_{B^{s+1}_{p,r}} + \|\rho_{k+1}\|_{B^s_{p,r}} \\
&\le \frac{2U_0 + C\int_0^t U_k^3 \mathrm{d}\tau + C\int_0^t \left(1-64CU_0^2\tau\right)^{\frac{1}{4}} U_k^3 \mathrm{d}\tau}{\left(1-64CU_0^2 t\right)^{\frac{1}{4}}} \\
&\le \frac{2U_0}{\left(1-64CU_0^2 t\right)^{\frac{1}{4}}} + \frac{2U_0}{\left(1-64CU_0^2 t\right)^{\frac{1}{4}}}\left(\frac{1}{\left(1-64CU_0^2 t\right)^{\frac{1}{4}}} - 1\right) \\
&\quad + \frac{4U_0}{\left(1-64CU_0^2 t\right)^{\frac{1}{4}}}\left(\frac{1}{\left(1-64CU_0^2 t\right)^{\frac{1}{4}}} - 1\right) \\
&\le \frac{4U_0}{\left(1-64CU_0^2 t\right)^{\frac{1}{4}}} + \frac{U_0}{\left(1-64CU_0^2 t\right)^{\frac{1}{4}}}\left(\frac{2}{\left(1-64CU_0^2 t\right)^{\frac{1}{4}}} - 4\right) \\
&\le \frac{4U_0}{\sqrt{1-64CU_0^2 t}},
\end{aligned}$$

in the last inequality we used that $t < T^* = \frac{3}{256CU_0^2}$.

Therefore, $(u_k, \rho_k)_{k\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}\left([0,T]; B^{s+1}_{p,r}\right) \times \mathcal{C}\left([0,T]; B^s_{p,r}\right)$. Using Equation (3.8) and the similar argument in the proof Lemma 3.1, one can easily prove that $(\partial_t u_k, \partial_t \rho_k)_{k\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}\left([0,T]; B^s_{p,r}\right) \times \mathcal{C}\left([0,T]; B^{s-1}_{p,r}\right)$, *i.e.* the sequence $(u_k, \rho_k)_{k\in\mathbb{N}}$ is uniformly bounded in $E^{s+1}_{p,r}(T) \times E^s_{p,r}(T)$.

Then, let us show that $(u_k, \rho_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}\left([0,T]; B^s_{p,r}\right) \times \mathcal{C}\left([0,T]; B^{s-1}_{p,r}\right)$. For all $k, j \in \mathbb{N}$, from (3.8), we can obtain that:

$$\begin{aligned}
\partial_t\left(u_{k+j+1} - u_{k+1}\right) = \Lambda^{-2}\partial_x\Big(& b_1\left(\rho_{k+j} - \rho_k\right) - \frac{b_2}{2}\left(u_{k+j} + u_k\right)\left(u_{k+j} - u_k\right) \\
&+ b_3\left(\rho_{k+j} - \rho_k\right)u_{k+j}^2 + b_3\rho_k\left(u_{k+j} + u_k\right)\left(u_{k+j} - u_k\right)\Big),
\end{aligned}$$

and

$$\partial_t\left(\rho_{k+j+1} - \rho_{k+1}\right) + \left(b_2 u_{k+j} - 2b_3\rho_{k+j}u_{k+j}\right)\partial_x\left(\rho_{k+j+1} - \rho_{k+1}\right) = G_{k,j},$$

with

$$\begin{aligned}
G_{k,j} = &-\left[b_2\left(u_{k+j} - u_k\right) - 2b_3\left(\left(\rho_{k+j} - \rho_k\right)u_{k+j} + \left(u_{k+j} - u_k\right)\rho_k\right)\right]\partial_x\rho_{k+1} \\
&+ b_4\partial_x\left(u_{k+j} - u_k\right) - b_2\left(\left(\rho_{k+j} - \rho_k\right)\partial_x u_{k+j} + \partial_x\left(u_{k+j} - u_k\right)\rho_k\right) \\
&+ b_3\left(\left(\rho_{k+j} - \rho_k\right)\left(\rho_{k+j} + \rho_k\right)\partial_x u_{k+j} + \partial_x\left(u_{k+j} - u_k\right)\rho_k^2\right).
\end{aligned}$$

Using the fact that $B^{s-1}_{p,r}$ is an algebra and the operator Λ is a $\mathcal{S}$-multiplier, and let $V^j_{k+1}(t) \doteq \|u_{k+j}-u_k\|_{B^s_{p,r}} + \|\rho_{k+j}-\rho_k\|_{B^{s-1}_{p,r}}$, for every $t\in[0,T^*)$, we have:

$$\|u_{k+j+1}-u_{k+1}\|_{B^s_{p,r}} \le \|u_{k+j+1}(0)-u_{k+1}(0)\|_{B^s_{p,r}} + C\int_0^t V^j_k(\tau) \cdot \left(1+\|u_k\|_{B^{s+1}_{p,r}} + \|u_{k+j}\|_{B^{s+1}_{p,r}} + \|\rho_k\|_{B^s_{p,r}} + \|\rho_{k+j}\|_{B^s_{p,r}}\right)^2 \mathrm{d}\tau,$$

moreover, using Lemma 2.6 again, we get:

$$\begin{aligned}\|\rho_{k+j+1}-\rho_{k+1}\|_{B^{s-1}_{p,r}} \le{}& \exp\left(C\int_0^t\left(\|u_{k+j}\|_{B^{s+1}_{p,r}} + \|u_{k+j}\|_{B^{s+1}_{p,r}}\|\rho_{k+j}\|_{B^s_{p,r}}\right)\mathrm{d}\tau\right)\\ &\cdot\|\rho_{k+j+1}(0)-\rho_{k+1}(0)\|_{B^{s-1}_{p,r}}\\ &+\int_0^t \exp\left(C\int_\tau^t\left(\|u_{k+j}\|_{B^{s+1}_{p,r}} + \|u_{k+j}\|_{B^{s+1}_{p,r}}\|\rho_{k+j}\|_{B^s_{p,r}}\right)\mathrm{d}\tau'\right)V^j_k(\tau)\\ &\cdot\left(1+\|u_k\|_{B^{s+1}_{p,r}} + \|u_{k+j}\|_{B^{s+1}_{p,r}} + \|\rho_k\|_{B^s_{p,r}} + \|\rho_{k+j}\|_{B^s_{p,r}}\right)^2 \mathrm{d}\tau.\end{aligned}$$

By using Proposition 2.1, we can obtain that:

$$\begin{aligned}\|\rho_{k+j+1}(0)-\rho_{k+1}(0)\|_{B^{s-1}_{p,r}} &= \|S_{k+j+1}\rho(0)-S_{k+1}\rho(0)\|_{B^{s-1}_{p,r}}\\ &=\left\|\sum_{d=k+1}^{k+j}\Delta_d u_0\right\|_{B^{s-1}_{p,r}}\\ &=\left(\sum_{k\ge -1}2^{k(s-1)\sigma}\left\|\Delta_k\left(\sum_{d=k+1}^{k+j}\Delta_d u_0\right)\right\|^\sigma_{L^p}\right)^{\frac{1}{\sigma}}\\ &\le C\left(\sum_{d=k}^{k+j+1}2^{-d\sigma}2^{d\sigma s}\|\Delta_d u_0\|^\sigma_{L^p}\right)^{\frac{1}{\sigma}}\\ &\le C2^{-k}\|\rho_0\|_{B^s_{p,r}},\end{aligned}$$

and

$$\|u_{k+j+1}(0)-u_{k+1}(0)\|_{B^s_{p,r}} \le C2^{-k}\|u_0\|_{B^{s+1}_{p,r}}.$$

In view of $(u_k,\rho_k)_{k\in\mathbb{N}}$ being uniformly bounded in $E^{s+1}_{p,r}(T)\times E^s_{p,r}(T)$, and combining the above inequalities, one may find a positive constant C_T independent of k, j such that:

$$V^j_{k+1}(t)\le C_T\left(2^{-k}+\int_0^t V^j_k(\tau)\mathrm{d}\tau\right),$$

for all $t\in[0,T)$. Moreover, using the induction procedure with respect to the index k, we have:

$$\begin{aligned}&\|u_{k+j+1}-u_{k+1}\|_{L^\infty_T(B^s_{p,r})} + \|\rho_{k+j+1}-\rho_{k+1}\|_{L^\infty_T(B^{s-1}_{p,r})}\\ &\le \frac{(TC_T)^{k+1}}{(k+1)!}\left(\|u_j\|_{L^\infty_T(B^{s+1}_{p,r})} + \|\rho_j\|_{L^\infty_T(B^s_{p,r})}\right) + \sum_{n=0}^k 2^{-(k-n)}\frac{(TC_T)^{n+1}}{(n+1)!}.\end{aligned}$$

Since that $\|u_j\|_{L^\infty_T(B^{s+1}_{p,r})}, \|\rho_j\|_{L^\infty_T(B^s_{p,r})}$ may be bounded independently of j, we conclude to the existence of some new constant C'_T independent of k, j such that:

$$\left\|u_{k+j+1}-u_{k+1}\right\|_{L_T^\infty\left(B_{p,r}^s\right)}+\left\|\rho_{k+j+1}-\rho_{k+1}\right\|_{L_T^\infty\left(B_{p,r}^{s-1}\right)}\le C_T' 2^{-k}.$$

Thus, we have proved that $(u_k,\rho_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}\left([0,T];B_{p,r}^s\right)\times\mathcal{C}\left([0,T];B_{p,r}^{s-1}\right)$. Hence, the proof of Lemma 3.2 is complete.

Finally, we prove the existence and uniqueness for (1.2) in Besov space.

Proof (Proof of Theorem 1.1) By using lemma 3.2, $(u_k,\rho_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}\left([0,T];B_{p,r}^s\right)\times\mathcal{C}\left([0,T];B_{p,r}^{s-1}\right)$, so it converges to some limit function $(u,\rho)\in\mathcal{C}\left([0,T];B_{p,r}^s\right)\times\mathcal{C}\left([0,T];B_{p,r}^{s-1}\right)$. Next, we have to prove that $(u,\rho)\in E_{p,r}^{s+1}(T)\times E_{p,r}^s(T)$ and solves (1.2). Using lemma 3.2 again, we can see that $(u_k,\rho_k)_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty\left([0,T];B_{p,r}^{s+1}\right)\times L^\infty\left([0,T];B_{p,r}^s\right)$. Fatou property for Besov spaces (Proposition 2.5 (6)) insures that (u,ρ) also belongs to $L^\infty\left([0,T];B_{p,r}^{s+1}\right)\times L^\infty\left([0,T];B_{p,r}^s\right)$.

On the other hand, as $(u_k,\rho_k)_{k\in\mathbb{N}}$ converges to (u,ρ) in $\mathcal{C}\left([0,T];B_{p,r}^s\right)\times\mathcal{C}\left([0,T];B_{p,r}^{s-1}\right)$, an interpolation argument guarantees that the convergence holds in $\mathcal{C}\left([0,T];B_{p,r}^{s'+1}\right)\times\mathcal{C}\left([0,T];B_{p,r}^{s'}\right)$, for any $s'<s$.

Passing limit in (3.8) reveals that (u,ρ) satisfy system (1.2). In view of the fact that (u,ρ) belongs to $L^\infty\left([0,T];B_{p,r}^{s+1}\right)\times L^\infty\left([0,T];B_{p,r}^s\right)$, for $s>1+\frac{1}{p}$, $B_{p,r}^{s+1}$ and $B_{p,r}^s$ are algebras, we obtain that the right-hand side of the equation:

$$u_t=\Lambda^{-2}\left(b_1\rho_x-b_2uu_x+b_3\left(\rho u^2\right)_x\right),$$

belongs to $L^\infty\left([0,T];B_{p,r}^{s+1}\right)$, and the right-hand side of the second equation:

$$\rho_t+(b_2-2b_3\rho)u\rho_x=b_4u_x-b_2\rho u_x+b_3\rho^2u_x,$$

belongs to $L^\infty\left([0,T];B_{p,r}^s\right)$.

In particular, for the case $r<\infty$, Lemma 2.6 implies that $(u,\rho)\in\mathcal{C}\left([0,T];B_{p,r}^{s'+1}\right)\times\mathcal{C}\left([0,T];B_{p,r}^{s'}\right)$ for any $s'<s$. Finally, by using the equation again, we see that $(\partial_t u,\partial_t\rho)\in\mathcal{C}\left([0,T];B_{p,r}^s\right)\times\mathcal{C}\left([0,T];B_{p,r}^{s-1}\right)$ if $r<\infty$, and in $L^\infty\left([0,T];B_{p,r}^s\right)\times L^\infty\left([0,T];B_{p,r}^{s-1}\right)$ otherwise. Therefore, the pair $(u,\rho)\in E_{p,r}^{s+1}(T)\times E_{p,r}^s(T)$.

The continuity with respect to the initial data in:

$$\left[\mathcal{C}\left([0,T];B_{p,r}^{s'+1}\right)\cap\mathcal{C}'\left([0,T];B_{p,r}^{s'}\right)\right]\times\left[\mathcal{C}\left([0,T];B_{p,r}^{s'-1}\right)\cap\mathcal{C}'\left([0,T];B_{p,r}^{s'-2}\right)\right],$$

for all $s'<s$, can be obtained by Lemma 3.1 and a simple interpolation argument. The case $s'=s$ can be proved through the use of a sequence of viscosity approximation solutions $(u_\varepsilon,\rho_\varepsilon)_{\varepsilon>0}$ for System (1.2) which converges uniformly in:

$$\left[\mathcal{C}\left([0,T];B_{p,r}^{s+1}\right)\cap\mathcal{C}'\left([0,T];B_{p,r}^{s}\right)\right]\times\left[\mathcal{C}\left([0,T];B_{p,r}^{s-1}\right)\cap\mathcal{C}'\left([0,T];B_{p,r}^{s-2}\right)\right],$$

gives the continuity of solution (u,ρ) in $E_{p,r}^{s+1}(T)\times E_{p,r}^s(T)$. Hence, the proof of Theorem 1.1 is complete.

4. Ill-Posedness in Besov Space

Next, we select the appropriate initial data to complete the proof of the Theorem 1.3. And we choose the initial data:

$$\rho_0(x)=\sum_{n=0}^{\infty}2^{-ns}\phi(x)\cos\left(\lambda 2^n x\right),$$

$$u_0(x)=\sum_{n=0}^{\infty}2^{-n(s+1)}\phi(x)\cos\left(\lambda 2^n x\right),$$

here $\lambda\in\left[\frac{67}{48},\frac{69}{48}\right]$ and $\hat{\phi}\in C_0^{\infty}(\mathbb{R})$ is an non-negative, even and real-valued function satisfying:

$$\hat{\phi}(x)=\begin{cases}1, & \text{if } |x|\le\frac{1}{4},\\ 0, & \text{if } |x|\ge\frac{1}{2}.\end{cases}$$

It is easy to prove that:

$$\operatorname{supp}\widehat{\phi(\cdot)\cos\left(\lambda 2^n\cdot\right)}\subset\left\{\xi:-\frac{1}{2}+\lambda 2^n\le|\xi|\le\frac{1}{2}+\lambda 2^n\right\},$$

then it can be verified for $j\ge 3$,

$$\Delta_j\left(\phi(x)\cos\left(\lambda 2^n x\right)\right)=\begin{cases}\phi(x)\cos\left(\lambda 2^n x\right), & j=n,\\ 0, & j\ne n,\end{cases}\tag{4.1}$$

so we have:

$$\begin{aligned}\|u_0\|_{B^{s+1}_{p,\infty}}&=\left\|\left(2^{j(s+1)}\|\Delta_j u_0\|_{L^p}\right)_{j\in\mathbb{Z}}\right\|_{l^\infty}\\&=\left\|\left(2^{j(s+1)}\left\|\Delta_j\sum_{n=0}^{\infty}2^{-n(s+1)}\phi(x)\cos\left(\lambda 2^n x\right)\right\|_{L^p}\right)_{j\in\mathbb{Z}}\right\|_{l^\infty}\\&=\left\|\left(2^{j(s+1)}\left\|2^{-j(s+1)}\phi(x)\cos\left(\lambda 2^j x\right)\right\|_{L^p}\right)_{j\in\mathbb{Z}}\right\|_{l^\infty}\\&\le\left\|\left\|\phi(x)\cos\left(\lambda 2^j x\right)\right\|_{L^p}\right\|_{l^\infty}\\&\le C.\end{aligned}$$

Similarly, we have:

$$\|\rho_0\|_{B^{s}_{p,\infty}}\le C,\|u_0\|_{B^{s+1}_{p,\infty}}\le C.\tag{4.2}$$

Next, we give several estimates that play an important role in the proof of the Theorem 1.3.

Lemma 4.1. Let $s>0$. Then, for the above constructed initial data (u_0,ρ_0), we have:

$$\|u_0\Delta_n\partial_x u_0\|_{L^p}\ge C2^{-ns},\tag{4.3}$$

and

$$\left\|u_0\Delta_n\partial_x\rho_0\right\|_{L^p}\geq C2^{-n(s-1)},\tag{4.4}$$

for n large enough.

Proof We only show (4.4), because (4.3) can be obtained in a similar way.

According to (4.1), we get:

$$\Delta_n\rho_0=\Delta_n\left(\sum_{n=0}^{\infty}2^{-ns}\phi(x)\cos\left(\lambda 2^n x\right)\right)=2^{-ns}\phi(x)\cos\left(\lambda 2^n x\right),$$

therefore, one has:

$$\begin{aligned}u_0\Delta_n\partial_x\rho_0&=u_0 2^{-ns}\partial_x\left(\phi(x)\cos\left(\lambda 2^n x\right)\right)\\&=u_0 2^{-ns}\left[\partial_x\phi\cos\left(\lambda 2^n x\right)-\phi(x)\sin\left(\lambda 2^n x\right)\lambda 2^n\right]\\&=2^{-ns}u_0\partial_x\phi\cos\left(\lambda 2^n x\right)-\lambda 2^{-n(s-1)}u_0\phi(x)\sin\left(\lambda 2^n x\right).\end{aligned}$$

Since $u_0(x)$ is a real valued continuous function on $\mathbb{R}$, then there exists $\sigma>0$, we have:

$$\left|u_0(x)\right|\geq\frac{1}{2}u_0(0)=\frac{1}{2}\sum_{n=0}^{\infty}2^{-n(s+1)}\phi(0)=\frac{2^s\phi(0)}{2^{s+1}-1},\tag{4.5}$$

for any $x\in B_\sigma(0)$.

Therefore, we derive from (4.5) that:

$$\begin{aligned}\left\|u_0\Delta_n\partial_x\rho_0\right\|_{L^p}&\geq C\left(2^{-n(s-1)}\left\|\phi(\cdot)\sin\left(\lambda 2^n\cdot\right)\right\|_{L^p(B_\sigma(0))}-2^{-ns}\left\|u_0\partial_x\phi\cos\left(\lambda 2^n x\right)\right\|_{L^p}\right)\\&\geq\left(C2^n-C_1\right)2^{-ns}.\end{aligned}$$

By choosing n large enough such that $C_1\leq C2^{n-1}$, then we can yield (4.4). So, Lemma 4.1 has been proved.

Lemma 4.2. Let $s>\max\left\{1+\frac{1}{p},\frac{3}{2}\right\}$. For the above constructed initial data (u_0,ρ_0), then there exists some $T=T\left(\left\|u_0\right\|_{B^{s+1}_{p,r}},\left\|\rho_0\right\|_{B^{s}_{p,r}}\right)$, for $0\leq t\leq T$, we have:

$$\left\|u(t)-u_0\right\|_{B^{s}_{p,\infty}}\leq Ct,\ \left\|\rho(t)-\rho_0\right\|_{B^{s-1}_{p,\infty}}\leq Ct.\tag{4.6}$$

Proof Since $(u_0,\rho_0)\in B^{s+1}_{p,\infty}\times B^{s}_{p,\infty}$ and according to the local existence result (see Theorem 1.1), hence the system (1.2) has a unique solution $(u,\rho)\in L^\infty\left([0,T];B^{s+1}_{p,\infty}\times B^{s}_{p,\infty}\right)$ for some $T=T\left(\left\|u_0\right\|_{B^{s+1}_{p,r}},\left\|\rho_0\right\|_{B^{s}_{p,r}}\right)$, and

$$\sup_{0\leq t\leq T}\left(\left\|u(t)\right\|_{B^{s+1}_{p,\infty}}+\left\|\rho(t)\right\|_{B^{s}_{p,\infty}}\right)\leq C\left(\left\|u_0\right\|_{B^{s+1}_{p,\infty}}+\left\|\rho_0\right\|_{B^{s}_{p,\infty}}\right).\tag{4.7}$$

Using the differential mean value theorem, the Minkowski inequality, Proposition 2.4 together with (4.7) for $t\in[0,T]$, we have:

$$\begin{aligned}\left\|\rho-\rho_0\right\|_{B^{s-1}_{p,\infty}}&\leq\int_0^t\left\|\partial_\tau\rho\right\|_{B^{s-1}_{p,\infty}}\mathrm{d}\tau\\&\leq C\int_0^t\left\|u\right\|_{B^{s}_{p,\infty}}+\left\|u\rho\right\|_{B^{s}_{p,\infty}}+\left\|\rho^2u\right\|_{B^{s}_{p,\infty}}\mathrm{d}\tau\\&\leq C\int_0^t\left\|u\right\|_{B^{s+1}_{p,\infty}}\left(1+\left\|\rho\right\|_{B^{s}_{p,\infty}}+\left\|\rho\right\|^2_{B^{s}_{p,\infty}}\right)\mathrm{d}\tau\\&\leq Ct,\end{aligned}$$

and

$$\left\|u-u_0\right\|_{B^s_{p,\infty}} \le C\int_0^t \left\|\rho\right\|_{B^{s-1}_{p,\infty}} + \left\|u^2\right\|_{B^{s-1}_{p,\infty}} + \left\|\rho u^2\right\|_{B^{s-1}_{p,\infty}} \mathrm{d}\tau \le Ct.$$

Therefore, we have completed the proof of Lemma 4.2.

Lemma 4.3. Under the assumption of Theorem 1.3, for all $0 \le t \le T$, we have:

$$\left\|\rho(t)-\rho_0-tv_0\right\|_{B^{s-2}_{p,\infty}} \le Ct^2, \left\|u(t)-u_0-tw_0\right\|_{B^{s-1}_{p,\infty}} \le Ct^2. \tag{4.8}$$

where

$$v_0 = b_4\partial_x u_0 - b_2\partial_x\left(\rho_0 u_0\right) + b_3\partial_x\left(\rho_0^2 u_0\right),$$

$$w_0 = \Lambda^{-2}\left(b_1\partial_x\rho_0 - b_2 u_0\partial_x u_0 + b_3\partial_x\left(\rho_0 u_0^2\right)\right).$$

Proof We denote that:

$$\begin{cases}\tilde{\rho} = \rho(t)-\rho_0-tv_0,\\ \tilde{u} = u(t)-u_0-tw_0.\end{cases}$$

Firstly, using the differential mean value theorem and the Minkowski inequality for $t\in[0,T]$, using Proposition 2.4, Proposition 2.5 and (4.7), we get:

$$\begin{aligned}\left\|\tilde{\rho}\right\|_{B^{s-2}_{p,\infty}} &\le \int_0^t \left\|\partial_\tau\rho - v_0\right\|_{B^{s-2}_{p,\infty}} \mathrm{d}\tau\\ &\le C\int_0^t \left\|u-u_0\right\|_{B^{s-1}_{p,\infty}} + \left\|u\rho-u_0\rho_0\right\|_{B^{s-1}_{p,\infty}} + \left\|u\rho^2-u_0\rho_0^2\right\|_{B^{s-1}_{p,\infty}} \mathrm{d}\tau\\ &\le C\int_0^t \tau\mathrm{d}\tau \le Ct^2,\end{aligned} \tag{4.9}$$

and

$$\left\|\tilde{u}\right\|_{B^{s-1}_{p,\infty}} \le \int_0^t \left\|\partial_\tau u - w_0\right\|_{B^{s-1}_{p,\infty}} \mathrm{d}\tau \le C\int_0^t \tau\mathrm{d}\tau \le Ct^2, \tag{4.10}$$

Thus, the proof of Lemma 4.3 is completed.

Proof (Proof of Theorem 1.3.) Based on the definition of the Besov norm, we have:

$$\begin{aligned}\left\|\rho-\rho_0\right\|_{B^s_{p,\infty}} &\ge 2^{ns}\left\|\Delta_n\left(\rho-\rho_0\right)\right\|_{L^p} = 2^{ns}\left\|\Delta_n\left(\tilde{\rho}+tv_0\right)\right\|_{L^p}\\ &\ge t2^{ns}\left\|\Delta_n v_0\right\|_{L^p} - 2^{ns}\left\|\Delta_n\tilde{\rho}\right\|_{L^p}\\ &\ge t2^{ns}\left\|\Delta_n\left(u_0\partial_x\rho_0\right)\right\|_{L^p} - Ct2^{ns}\left\|\Delta_n\rho_0\partial_x u_0\right\|_{L^p} - 2^{ns}\left\|\Delta_n\tilde{\rho}\right\|_{L^p}\\ &\ge t2^{ns}\left\|\Delta_n\left(u_0\partial_x\rho_0\right)\right\|_{L^p} - Ct\left\|\rho_0\partial_x u_0\right\|_{B^s_{p,\infty}} - C2^{2n}\left\|\tilde{\rho}\right\|_{B^{s-2}_{p,\infty}}.\end{aligned} \tag{4.11}$$

Since

$$\begin{aligned}\Delta_n\left(u_0\partial_x\rho_0\right) &= \Delta_n\left(u_0\partial_x\rho_0\right) - u_0\Delta_n\left(\partial_x\rho_0\right) + u_0\Delta_n\left(\partial_x\rho_0\right)\\ &= \left[\Delta_n, u_0\right]\partial_x\rho_0 + u_0\Delta_n\left(\partial_x\rho_0\right),\end{aligned}$$

Using Proposition 2.5, Lemmas 2.8 and (4.2), we deduce that:

$$\left\|\rho_0\partial_x u_0\right\|_{B^s_{p,\infty}} \le C\left\|\rho_0\right\|_{B^s_{p,\infty}}\left\|u_0\right\|_{B^{s+1}_{p,\infty}} \le C,$$

and

$$\left\|2^{ns}\left\|\left[\Delta_n, u_0\right]\partial_x\rho_0\right\|_{L^p}\right\|_{l^\infty} \le C\left(\left\|\partial_x u_0\right\|_{L^\infty}\left\|\rho_0\right\|_{B^s_{p,\infty}} + \left\|\partial_x\rho_0\right\|_{L^\infty}\left\|u_0\right\|_{B^s_{p,\infty}}\right) \le C.$$

Taking above estimates into (4.11), we have:

$$\|\rho-\rho_0\|_{B^s_{p,\infty}} \ge t2^{ns}\|u_0\Delta_n(\partial_x\rho_0)\|_{L^p} - C_1t - C_2 2^{2n}\|\tilde{\rho}\|_{B^{s-2}_{p,\infty}}.$$

And using Lemma 4.1 and Lemma 4.3, we have:

$$\|\rho-\rho_0\|_{B^s_{p,\infty}} \ge C_3t2^{ns}2^{-n(s-1)} - C_1t - C_2 2^{2n}t^2 \ge C_3t2^n - C_1t - C_2 2^{2n}t^2.$$

Choosing n large enough such that $C_3 2^n > 2C_1$, then we get:

$$\|\rho-\rho_0\|_{B^s_{p,\infty}} \ge \frac{C_3t2^n}{2} - C_2 2^{2n}t^2.$$

Picking $t2^n \approx \delta < \frac{C_3}{4C_2}$ when $t \to 0$, we have:

$$\|\rho-\rho_0\|_{B^s_{p,\infty}} \ge \frac{C_3}{2}\delta - C_2\delta^2 \ge \frac{C_3}{4}\delta.$$

Likewise, we have:

$$\|u-u_0\|_{B^{s+1}_{p,\infty}} \ge \frac{C_4}{4}\delta.$$

This completes the proof of Theorem 1.3.

5. Gevrey Regularity for System (1.2)

In this section, we will apply nonlinear Cauchy-Kowalevski theory to establish the existence of analytic solutions to system (1.2). To this purpose, it is necessary to introduce the Cauchy-Kowalevski theorem.

Theorem 5.1. (See [11] [12]) Let $\left(X_\delta, \|\cdot\|_\delta\right)_{0<\delta\le 1}$ be a scale of decreasing Banach spaces, such that for any $0<\delta'<\delta\le 1$, we have $X_\delta \subset X_{\delta'}$ with $\|\cdot\|_{\delta'} < \|\cdot\|_\delta$. Consider the Cauchy problem:

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = F(t,u(t)), \\ u(0) = 0. \end{cases} \tag{5.1}$$

Let $T, R > 0$ and $\sigma \ge 1$. For given $u_0 \in X_1$, assume that F satisfies the following conditions:

(H_1) If for any $0<\delta'<\delta<1$, the function $t \mapsto u(t)$ is holomorphic on $|t|<T$ and continuous on $|t|\le T$ with values in X_δ and

$$\sup_{|t|<T}\|u(t)\|_\delta < R,$$

then $t \mapsto F(t,u(t))$ is a holomorphic function on $|t|<T$ with values in $X_{\delta'}$.

(H_2) For any $0<\delta'<\delta<1$ and $u,v \in \overline{B(u_0,R)} \subset X_\delta$, there exists a positive constant L depending on u_0 and R such that:

$$\sup_{|t|<T}\|F(t,u)-F(t,v)\|_{\delta'} \le \frac{L}{(\delta-\delta')^\sigma}\|u-v\|_\delta.$$

(H_3) There exists a $M>0$ depending on u_0 and R such that for any $0<\delta<1$,

$$\sup_{|t|<T}\left\|F(t,0)\right\|_{\delta}\le\frac{M}{(1-\delta)^{\sigma}}.$$

Then, there exists a $T_0\in(0,T)$ and a unique function $u(t)$ to the Cauchy problem (5.1), which is holomorphic in $|t|<\frac{(1-\delta)^{\sigma}T_0}{2^{\sigma}-1}$ with values in X_{δ} for every $\delta\in(0,1)$.

We would like to include four properties of the spaces $G^{\delta}_{\sigma,s}$, which will be used in the proof of Theorem 1.5 (the proofs of these properties can be find in [12]).

Proposition 5.2. Let $0<\delta'<\delta$, $0<\sigma'<\sigma$ and $s'<s$. From Definition 1.3, one can check that $G^{\delta}_{\sigma,s}\hookrightarrow G^{\delta'}_{\sigma,s}$, $G^{\delta}_{\sigma',s}\hookrightarrow G^{\delta}_{\sigma,s}$ and $G^{\delta}_{\sigma,s}\hookrightarrow G^{\delta}_{\sigma,s'}$.

Proposition 5.3. Let s be a real number and $\sigma>0$. Assume that $0<\delta'<\delta$. Then, we have:

$$\left\|\partial_x f\right\|_{G^{\delta'}_{\sigma,s}}\le\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\left\|f\right\|_{G^{\delta}_{\sigma,s}}. \tag{5.2}$$

Proposition 5.4. (Product acts on Sobolev-Gevrey spaces with $d=1$) Let $s>\frac{1}{2},\sigma\ge1$ and $\delta>0$. Then, $G^{\delta}_{\sigma,s}$ is an algebra. Moreover, there exists a constant $\overline{C_s}$ such that:

$$\left\|fg\right\|_{G^{\delta}_{\sigma,s}}\le\overline{C_s}\left\|f\right\|_{G^{\delta}_{\sigma,s}}\left\|g\right\|_{G^{\delta}_{\sigma,s}}. \tag{5.3}$$

Proposition 5.5. Let $s>\frac{1}{2},\sigma\ge1$ and $\delta>0$. There exists a constant C_s such that"

$$\left\|fg\right\|_{G^{\delta}_{\sigma,s-1}}\le C_s\left\|f\right\|_{G^{\delta}_{\sigma,s-1}}\left\|g\right\|_{G^{\delta}_{\sigma,s}}. \tag{5.4}$$

Now, we use all the above tools to prove Theorem 1.5.

Proof (Proof of Theorem 1.5) Assume that:

$$F(z)\doteq\begin{pmatrix}F_1(z)\\F_2(z)\end{pmatrix}=\begin{pmatrix}\Lambda^{-2}\left(b_1\rho_x-b_2uu_x+b_3u^2\rho_x+2b_3\rho uu_x\right)\\ b_4u_x-b_2\rho u_x-b_2u\rho_x+b_3\rho^2u_x+2b_3\rho u\rho_x\end{pmatrix}.$$

By using Definition 1.3 and Proposition 5.2, we can see that $G^{\delta}_{\sigma',s}$ is a scale of decreasing Banach spaces for a fixed $\sigma\ge1$, $s>\frac{1}{2}$ and $0<\delta\le1$. Moreover, for $0<\delta'<\delta$, $\sigma\ge1$, $s>\frac{1}{2}$ and the estimates (5.2)-(5.4), we have:

$$\begin{aligned}\left\|F_2(u,\rho)\right\|_{G^{\delta'}_{\sigma,s}}&\le\left\|b_4u_x-b_2\rho u_x-b_2u\rho_x+b_3\rho^2u_x+2b_3\rho u\rho_x\right\|_{G^{\delta'}_{\sigma,s}}\\&\le C_{b_2,b_3,b_4}\left(\left\|u_x\right\|_{G^{\delta'}_{\sigma,s}}+\left\|\rho u_x\right\|_{G^{\delta'}_{\sigma,s}}+\left\|u\rho_x\right\|_{G^{\delta'}_{\sigma,s}}+\left\|\rho^2u_x\right\|_{G^{\delta'}_{\sigma,s}}+\left\|\rho u\rho_x\right\|_{G^{\delta'}_{\sigma,s}}\right)\\&\le C_{s,b_2,b_3,b_4}\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\left(\left\|\rho\right\|_{G^{\delta}_{\sigma,s}}+\left\|u\right\|_{G^{\delta}_{\sigma,s}}\right)\left(\left\|\rho\right\|_{G^{\delta}_{\sigma,s}}+\left\|u\right\|_{G^{\delta}_{\sigma,s}}+1\right)^2,\end{aligned}$$

and

$$\begin{aligned}\|F_1(u,\rho)\|_{G^{\delta'}_{\sigma,s}} &\le \|b_1\rho_x - b_2uu_x + b_3u^2\rho_x + 2b_3\rho uu_x\|_{G^{\delta'}_{\sigma,s-2}} \\ &\le C_{s,b_1,b_2,b_3}\frac{e^{-\sigma}\sigma^\sigma}{(\delta-\delta')^\sigma}\left(\|\rho\|_{G^{\delta}_{\sigma,s}} + \|u\|_{G^{\delta}_{\sigma,s}}\right)\left(\|\rho\|_{G^{\delta}_{\sigma,s}} + \|u\|_{G^{\delta}_{\sigma,s}} + 1\right)^2,\end{aligned}$$

where C_{s,b_1,b_2,b_3,b_4} is a constant depends only on s,b_1,b_2,b_3,b_4. Therefore, we can obtain that:

$$\begin{aligned}&\|F_1(u,\rho)\|_{G^{\delta'}_{\sigma,s}} + \|F_2(u,\rho)\|_{G^{\delta'}_{\sigma,s}} \\ &\le C_{s,b_1,b_2,b_3,b_4}\frac{e^{-\sigma}\sigma^\sigma}{(\delta-\delta')^\sigma}\left(\|\rho\|_{G^{\delta}_{\sigma,s}} + \|u\|_{G^{\delta}_{\sigma,s}}\right)\left(\|\rho\|_{G^{\delta}_{\sigma,s}} + \|u\|_{G^{\delta}_{\sigma,s}} + 1\right)^2,\end{aligned}$$

Similarly, for any $0<\delta<1$, we also can obtain that:

$$\begin{aligned}&\|F_1(u_0,\rho_0)\|_{G^{\delta}_{\sigma,s}} + \|F_2(u_0,\rho_0)\|_{G^{\delta}_{\sigma,s}} \\ &\le C_{s,b_1,b_2,b_3,b_4}\frac{e^{-\sigma}\sigma^\sigma}{(1-\delta)^\sigma}\left(\|\rho_0\|_{G^{1}_{\sigma,s}} + \|u_0\|_{G^{1}_{\sigma,s}}\right)\left(\|\rho_0\|_{G^{1}_{\sigma,s}} + \|u_0\|_{G^{1}_{\sigma,s}} + 1\right)^2.\end{aligned}$$

Therefore, these spaces and $F(u,\rho)$ satisfy condition (H_1) and (H_3) in Theorem 5.1.

Next, in order to prove our desire result, it suffices to show that $F(u,\rho)$ satisfies the condition (H_2) in Theorem 5.1. Assume that $\|z-z_0\|_\delta \le R$ and $\|z'-z_0'\|_\delta \le R$ with $\|z\|_\delta = \|\rho\|_{G^{\delta}_{\sigma,s}} + \|u\|_{G^{\delta}_{\sigma,s}}$, for $0<\delta'<\delta$, $s>\frac{1}{2}$, we can see that:

$$\begin{aligned}&\|F_2(z)-F_2(z')\|_{G^{\delta'}_{\sigma,s}} \\ &\le \|b_4u_x - b_2\rho u_x - b_2u\rho_x + b_3\rho^2u_x + 2b_3\rho u\rho_x\|_{G^{\delta'}_{\sigma,s}} \\ &\le C_{s,b_2,b_3,b_4}\Big(\|u_x-u_x'\|_{G^{\delta'}_{\sigma,s}} + \|(\rho-\rho')u_x\|_{G^{\delta'}_{\sigma,s}} + \|\rho'(u_x-u_x')\|_{G^{\delta'}_{\sigma,s}} \\ &\quad + \|(u-u')\rho_x\|_{G^{\delta'}_{\sigma,s}} + \|u'(\rho_x-\rho_x')\|_{G^{\delta'}_{\sigma,s}} + \left\|\left(\rho^2-(\rho')^2\right)u_x\right\|_{G^{\delta'}_{\sigma,s}} \\ &\quad + \left\|(\rho')^2(u_x-u_x')\right\|_{G^{\delta'}_{\sigma,s}} + \|(\rho u-\rho'u')\rho_x\|_{G^{\delta'}_{\sigma,s}} + \|\rho'u'(\rho_x-\rho_x')\|_{G^{\delta'}_{\sigma,s}}\Big) \\ &\le C_{s,b_2,b_3,b_4}\frac{e^{-\sigma}\sigma^\sigma}{(\delta-\delta')^\sigma}\left(\|u-u'\|_{G^{\delta}_{\sigma,s}} + \|\rho-\rho'\|_{G^{\delta}_{\sigma,s}}\right) \\ &\quad\left(\|\rho\|_{G^{\delta}_{\sigma,s}} + \|\rho'\|_{G^{\delta}_{\sigma,s}} + \|u\|_{G^{\delta}_{\sigma,s}} + \|u'\|_{G^{\delta}_{\sigma,s}} + 1\right)^2,\end{aligned}$$

and

$$\begin{aligned}\|F_1(z)-F_1(z')\|_{G^{\delta'}_{\sigma,s}} &\le C_{s,b_1,b_2,b_3}\frac{e^{-\sigma}\sigma^\sigma}{(\delta-\delta')^\sigma}\left(\|u-u'\|_{G^{\delta}_{\sigma,s}} + \|\rho-\rho'\|_{G^{\delta}_{\sigma,s}}\right) \\ &\quad\left(\|\rho\|_{G^{\delta}_{\sigma,s}} + \|\rho'\|_{G^{\delta}_{\sigma,s}} + \|u\|_{G^{\delta}_{\sigma,s}} + \|u'\|_{G^{\delta}_{\sigma,s}} + 1\right)^2.\end{aligned}$$

Thus,

$$
\begin{aligned}
&\left\|F(z)-F(z')\right\|_{G^{\delta'}_{\sigma,s}} \\
&\le \left\|F_1(z)-F_1(z')\right\|_{G^{\delta'}_{\sigma,s}} + \left\|F_2(z)-F_2(z')\right\|_{G^{\delta'}_{\sigma,s}} \\
&\le C_{s,b_1,b_2,b_3,b_4}\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\left(\left\|u-u'\right\|_{G^{\delta}_{\sigma,s}}+\left\|\rho-\rho'\right\|_{G^{\delta}_{\sigma,s}}\right) \\
&\quad \left(\left\|\rho\right\|_{G^{\delta}_{\sigma,s}}+\left\|\rho'\right\|_{G^{\delta}_{\sigma,s}}+\left\|u\right\|_{G^{\delta}_{\sigma,s}}+\left\|u'\right\|_{G^{\delta}_{\sigma,s}}+1\right)^2 \\
&\le C_{s,b_1,b_2,b_3,b_4}\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\left(\left\|z-z'\right\|_{\delta}\right)\left(\left\|z\right\|_{\delta}+\left\|z'\right\|_{\delta}+1\right)^2 \\
&\le C_{s,b_1,b_2,b_3,b_4}\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\left(\left\|z-z'\right\|_{\delta}\right)\left(2\left\|z_0\right\|_{\delta}+2R+1\right)^2,
\end{aligned}
$$

where in the last inequality we apply the fact that $\left\|z-z_0\right\|_{\delta}\le R$ and $\left\|z'-z'_0\right\|_{\delta}\le R$. From the above inequality, for $0<\delta'<\delta$, $\sigma\ge 1$, $s>\frac{1}{2}$, we verify that $F(z)$ satisfies the condition (H_2) of Theorem 5.1 with $L=C_{s,b_1,b_2,b_3,b_4}e^{-\sigma}\sigma^{\sigma}\left(2\left\|z_0\right\|_{\delta}+2R+1\right)^2$. This completes the proof of Theorem 1.5.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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