Hindawi
Abstract and Applied Analysis
Volume 2022, Article ID 7688168, 13 pages
https://doi.org/10.1155/2022/7688168

Research Article

Coincidence Fixed-Point Theorems for p-Hybrid Contraction Mappings in G_b-Metric Space with Application

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Received 25 July 2022; Revised 30 August 2022; Accepted 24 November 2022; Published 8 December 2022

Academic Editor: Paul Eloe

By combining the notions of G-metric space and b-metric space, in this paper, we present coincidence fixed-point theorems for p-hybrid mappings in G_b-metric spaces. An example is given to demonstrate the novelty of our main results. Henceforth, the illustrative applications are given by using nonlinear fractional differential equations.

1. Introduction

In 1922, Banach [1] initiated the study of constructive theory in metric space. The constructive theory is used for nonlinear functional analysis, approximation theory, optimization theory (saddle function), variation inequalities, game theory (Nash equilibrium), and economics (Black-Scholes theorem). In addition, it is used in many practical and research problems in various fields beyond mathematics. It can reduce to fixed-point problems, which include biology, chemistry, physics, computer science, economics, engineering, global analysis statistics, and operations research.

In 1969, Nadler [2] proved the multivalued version of Banach's contraction principle [1]. Naimpally et al. [3] generalized Goebel's [4] result in a hybrid contraction mapping. The method of hybrid fixed points can be used to derive another classical fixed-point theorem result. The concept of hybrid pair of mapping is very consequential for the theory of fixed point, and it has an important role in game theory, optimization theory, and differential equations.

Definition 1 (see [3]). Let $\mathscr{P} : \mathscr{W} \longrightarrow \mathscr{CL}(\mathscr{W})$ and $\mathscr{Q} : \mathscr{W} \longrightarrow \mathscr{W}$ be such that $\mathscr{P}\mathscr{W}$ is a complete subspace of $\mathscr{W}$ and $\mathscr{Q}\mathscr{W} \subset \mathscr{P}\mathscr{W}$. Further, assume there exists θ, $0 < \theta < 1$ such that for every $\kappa, \mu \in \mathscr{W}$,

$$\mathscr{H}(\mathscr{P}\kappa, \mathscr{P}\mu) \leq \theta d(\mathscr{Q}\kappa, \mathscr{Q}\mu). \tag{1}$$

Then, $\mathscr{P}$ and $\mathscr{Q}$ have a coincidence; that is, there exists $\nu \in \mathscr{W}$ such that $\mathscr{Q}\nu \in \mathscr{P}\nu$.

Later, Chauhan et al. [5] proved the results in unified common fixed-point theorems for a hybrid pair of mappings via an implicit relation involving altering distance function. Imdad et al. [6, 7] generalized the hybrid fixed-point theorems in symmetric spaces via common limit range property and joint common limit range property in metric spaces. Nashine et al. [8] gave the proof using (JCLR) property for hybrid fixed-point theorems via quasi F-contractions. Wangwe and Kumar [9, 10] proved the common fixed-point theorem for a hybrid pair of mappings in weak partial b-metric spaces and G-metric space with some applications.

Bakhtin [11] and Czerwik [12] generalized the concept of metric space to b-metric space due to some problems, especially the convergence issue of measurable functions to a measure that led to a generalization of a metric's notion. Czerwik [12] established b-metric spaces by introducing a parameter $s \geq 1$ in the triangle inequality as a coefficient and generalized Banach's contraction principle to these spaces. Later, Czerwik [13] proved the multivalued results in b-metric spaces. These findings motivated several potential researchers to perform and analyze contraction condition variants using single- and multivalued maps in b-metric space. One can refer to [14–18] and the references therein.

The concepts of G-metric space were initiated by Mustafa and Sims [19, 20] due to the shortcoming of the fundamental topological structure on D-metric spaces. Also, they replaced the tetrahedral inequality with an inequality involving the repetition of indices. Further, several researchers generalized the results for single-valued mapping and multivalued mappings in G-metric spaces. For more results, we refer the reader in [21–25] and the references contained. Furthermore, Aghajani et al. [26] using both concepts of b-metrics and G-metrics initiated the results on G_b-metric spaces. Since then, several results followed for single- and multivalued mappings for various abstract spaces. For more literature, we refer the reader to [27–32] and the references contained.

This paper is aimed at proving a coincidence fixed-point theorem for p-hybrid contraction mappings in G_b-metric space with some application to the fractional differential equation. In particular, we modify and extend the works due to Karapinar et al. [33, 34], Wangwe and Kumar [10], and Aghajani et al. [26]. The results proved to have a novelty in the study of fixed-point theory.

2. Preliminaries

This part introduces some preliminary results of definitions and theorems, which will help develop the main result.

Bakhtin [11] and Czerwik [12] established a new metric on a nonempty set, known as a b-metric.

Definition 2 (see [12]). Let $\mathscr{W}$ be a nonempty set and $s \geq 1$ be a given real number. Suppose that a function $d_b : \mathscr{W} \times \mathscr{W} \longrightarrow [0,\infty)$ satisfies

(B1) $d_b(\kappa, \mu) = 0$ iff $\kappa = \mu$

(B2) $d_b(\kappa, \mu) = d_b(\mu, \kappa)$

(B3) $d_b(\kappa, \mu) \leq s[d_b(\kappa, \nu) + d_b(\nu, \kappa)]$, for all $\kappa, \mu, \nu \in \mathscr{W}$

Then, d_b is said to be b-metric and $(\mathscr{W}, d_b, s)$ is a b-metric space. $s \geq 1$ is defined as a parameter of $(\mathscr{W}, d_b, s)$.

We give some examples which satisfy b-metric space axioms.

Example 1 (see [11, 12]). Consider the set $\mathscr{W} = [0, 1]$ endowed with the function $d_b : \mathscr{W} \times \mathscr{W} \longrightarrow [0,\infty)$ defined by $d_b(\kappa, \mu) = |\kappa - \mu|^2$ for all $\kappa, \mu \in \mathscr{W}$. Thus, $(\mathscr{W}, d_b, 2)$ is a b-metric space for $s = 2$.

Mustafa and Sims [20] gave the following axioms for G-metric space.

Definition 3 (see [20]). Let $\mathscr{W}$ represent a nonempty set with $G : \mathscr{W} \times \mathscr{W} \times \mathscr{W} \longrightarrow \mathbb{R}_+$ as a function which satisfies the following axioms:

(G1) $G(\kappa, \mu, \nu) = 0$ for $\kappa = \mu = \nu$

(G2) $G(\kappa, \kappa, \mu) > 0$, $\forall\ \kappa, \mu \in \mathscr{W}$ if $\nu{=}\mu$

(G3) $G(\kappa, \kappa, \mu) \leq G(\kappa, \mu, \nu)$ $\forall\ \kappa, \mu \in \mathscr{W}$ if $\nu{=}\mu$

(G4) $G(\kappa, \mu, \nu) = G(\kappa, \nu, \mu) = G(\mu, \nu, \kappa) = \cdots$

(G5) $G(\kappa, \mu, \nu) \leq G(\kappa, a, a) + G(a, \mu, \nu)$, $\forall\ \kappa, \mu, \nu, a \in \mathscr{W}$

Then, G is called a metric and $(\mathscr{W}, G)$ is a G-metric space.

We give an example from [20].

Example 2 (see [20]). Let $\mathscr{W} = \mathbb{R}$ represent a set of real numbers. Define $G : \mathbb{R}^3 \longrightarrow \mathbb{R}^+$ as follows:

$$G(\kappa, \mu, \nu) = |\kappa - \mu| + |\mu - \nu| + |\kappa - \nu|, \tag{2}$$

for all $\kappa, \mu, \nu \in \mathscr{W}$.

Recall that if $G(\kappa, \mu, \nu) = 0$, then $\kappa = \mu = \nu$.

Example 3 (see [20]). Let $\mathscr{W} = \mathbb{R}$. Then, G-metric G is defined by

$$G(\kappa, \mu, \nu) = \frac{1}{3}(|\kappa - \mu| + |\mu - \nu| + |\kappa - \nu|), \tag{3}$$

for all $\kappa, \mu, \nu \in \mathscr{W}$.

On the other hand, Aghajani et al. combining the concepts from Bakhtin [11], Czerwik [12] and Mustafa and Sims [20], they established a new generalized space known as G_b-metric space.

Definition 4 (see [26]). Let $\mathscr{W}$ be a nonempty set and $s \geq 1$. Suppose that $G_b : \mathscr{W} \times \mathscr{W} \times \mathscr{W} \longrightarrow [0,\infty)$ is a function satisfying the following conditions:

(G_b1) $G_b(\kappa, \mu, \nu) = 0$ if $\kappa = \mu = \nu$

(G_b2) $0 < G(\kappa, \kappa, \mu)$, for all $\kappa, \mu \in \mathscr{W}$ with $\nu{=}\mu$

(G_b3) $G(\kappa, \kappa, \mu) \leq G(\kappa, \mu, \nu)$ for all $\kappa, \mu \in \mathscr{W}$ with $\nu{=}\mu$

(G_b4) $G(\kappa, \mu, \nu) = G(\kappa, \nu, \mu) = G(\mu, \nu, \kappa) = \cdots$ (symmetry in all three variables)

(G_b5) $G(\kappa, \mu, \nu) \leq s[G(\kappa, a, a) + G(a, \mu, \nu)]$, for all $\kappa, \mu, \nu, a \in \mathscr{W}$ (rectangle inequality)

The distance metric G_b is called a G_b-metric, and $(\mathscr{W}, G_b)$ is called a G_b-metric space. The real number $s \geq 1$ is called the coefficient of $(\mathscr{W}, G_b)$.

Let $\mathscr{W} = \mathbb{R}$ be the set of real numbers; then, the mapping $G_b : \mathscr{W} \times \mathscr{W} \times \mathscr{W} \longrightarrow [0,\infty)$ is defined by

$$G_b(\kappa, \mu, \nu) = \frac{1}{9}[|\kappa - \mu| + |\mu - \nu| + |\kappa - \nu|]^2, \tag{4}$$

for all $\kappa, \mu, \nu \in \mathbb{R}$, is a G_b-metric.

Example 4 (see [26]). Let $(\mathscr{W}, G_b)$ be a G_b-metric space. Consider

$$G_b(\kappa, \mu, \nu) = (G_b(\kappa, \mu, \nu))^p, \tag{5}$$

for all $\kappa, \mu, \nu \in \mathscr{W}$ and $p > 1$. Therefore, G_b is a G_b-metric with $s = 2^{p-1}$.

Example 5 (see [26]). Let $(\mathscr{W}, G_b)$ be a G_b-metric space. Then, for $\kappa_0 \in \mathscr{W}$, $r > 0$, the G_b-ball with center κ_0 and radius r is

$$B_G(\kappa_0, r) = \{\mu \in \mathscr{W} | G(\kappa_0, \mu, \mu) < r\}. \tag{6}$$

For example, $\kappa = \mathbb{R}$, and consider a G_b-metric G defined by

$$G_b(\kappa, \mu, \nu) = \frac{1}{9}\left[|\kappa - \mu| + |\mu - \nu| + |\kappa - \nu|\right]^2, \tag{7}$$

for all $\kappa, \mu, \nu \in \mathbb{R}$. Then,

$$\begin{aligned} B_G(3,4) &= \{\mu \in \mathscr{W} \mid G(3,\mu,\mu) < 4\} \\ &= \left\{\mu \in \mathscr{W} \mid \frac{1}{9}\left[|\kappa - \mu| + |\mu - \nu| + |\kappa - \nu|\right]^2 < 4\right\} \\ &= \left\{\mu \in \mathscr{W} \mid \frac{1}{9}\left[|3 - \mu| + |\mu - \nu| + |3 - \nu|\right]^2 < 4\right\} \\ &= \left\{\mu \in \mathscr{W} \mid \frac{1}{9}\left[|3 - \mu| + |3 - \nu|\right]^2 < 4\right\} \\ &= \left\{\mu \in \mathscr{W} \mid \frac{4}{9}|3 - \mu|^2 < 4\right\} \\ &= \left\{\mu \in \mathscr{W} \mid |3 - \mu|^2 < 9\right\} = (0, 6). \end{aligned} \tag{8}$$

Aghajani et al. [26] gave the proposition below that satisfies G_b-metric axioms.

Proposition 5 (see [26]). *Let $(\mathscr{W}, G_b)$ be a G_b-metric space, $\forall\ \kappa, \mu, \nu, a \in \mathscr{W}$; we have the following:*

(i) If $G_b(\kappa, \mu, \nu) \le G_b(\kappa, \kappa, \mu) + G_b(\kappa, \kappa, \nu)$, then $\kappa = \mu = \nu$

(ii) $G_b(\kappa, \mu, \nu) \le s[G_b(\kappa, \kappa, \mu) + G_b(\kappa, \kappa, \nu)]$

(iii) $G_b(\kappa, \mu, \mu) \le 2sG_b(\mu, \kappa, \kappa)$

(iv) $G_b(\kappa, \mu, \nu) \le s[G_b(\kappa, a, \nu) + G_b(a, \mu, \nu)]$

Motivated by Aghajani et al. [26], we recall some properties in G_b-metric spaces as follows.

Definition 6 (see [26]). Let $(\mathscr{W}, G_b)$ be a G_b-metric space. If $\{\kappa_i\} \in \mathscr{W}$ then, we have the following:

(i) G_b is convergence to a point $\kappa \in \mathscr{W}$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all i, $j \ge n_0$, $G_b(\kappa_i, \kappa_j, \kappa) < \varepsilon$

(ii) G_b is a Cauchy sequence if, for $\varepsilon > 0$, $\exists$ a positive integer n_0 such that, for all $i, j, k \ge n_0$, $G_b(\kappa_i, \kappa_j, \kappa_k) < \varepsilon$

Proposition 7 (see [26, 27]). *Let $(\mathscr{W}, G_b)$ be a G_b-metric space. Then, the function is given by*

$$d_{G_b}(\kappa, \mu) = G_b(\kappa, \mu, \mu) + G_b(\mu, \kappa, \kappa), \tag{9}$$

for all $\kappa, \mu \in \mathscr{W}$. Define a b-metric on $\mathscr{W}$. It is called a b-metric induced by the G_b-metric G.

Proposition 8 (see [27]). *Let $(\mathscr{W}, G_b)$ be a G_b-metric space. The properties below are similar:*

(i) $\{\kappa_i\}$ is G_b-convergence to κ

(ii) $\lim_{i \longrightarrow \infty} G_b(\kappa_i, \kappa_i, \kappa) = 0$

(iii) $\lim_{i \longrightarrow \infty} G_b(\kappa_i, \kappa, \kappa) = 0$

(iv) $\lim_{i \longrightarrow \infty} G_b(\kappa_i, \kappa_j, \kappa_k) = 0$

Proposition 9 (see [27]). *Let $(\mathscr{W}, G_b)$ be a G_b-metric space. Therefore, the properties below hold:*

(i) $\{\kappa_i\}$ is a G_b-Cauchy sequence

(ii) $\forall\ \varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, $\forall i, j \in n_0$, $G(\kappa_i, \kappa_j, \kappa_k) = 0$

Definition 10 (see [26]). A G_b-metric space $\mathscr{W}$ is called G_b-complete if every G_b Cauchy sequence is a G_b-convergent in $\mathscr{W}$.

Further, Makran et al. [29] extended the works due to Aghajani et al. [26] and Kaewcharoen and Kaewkhao [22] by introducing the multivalued versions in G_b-metric spaces as follows:

Let $\mathscr{W}$ be a G_b-metric space. We shall denote $\mathscr{CB}^{G_b}(\mathscr{W})$ as a nonempty, closed, and bounded subsets of $\mathscr{W}$. Let $\mathscr{H}_{G_b}(.,.,.)$ represent the Haursdorff-G_b-metric on $\mathscr{CB}^{G_b}(\mathscr{W})$, and $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathscr{CB}^{G_b}(\mathscr{W})$ define

$$\mathscr{H}_{G_b}(\mathscr{A}, \mathscr{B}, \mathscr{C}) = \max\left\{\sup_{\kappa \in \mathscr{A}} G_b(\kappa, \mathscr{B}, \mathscr{C}), \sup_{\kappa \in \mathscr{B}} G_b(\kappa, \mathscr{C}, \mathscr{A}), \sup_{\kappa \in \mathscr{C}} G_b(\kappa, \mathscr{A}, \mathscr{B})\right\}, \tag{10}$$

where

$$G_b(\kappa, \mathscr{B}, \mathscr{C}) = d_{G_b}(\kappa, \mathscr{B}) + d_{G_b}(\mathscr{B}, \mathscr{C}) + d_{G_b}(\kappa, \mathscr{C}), \tag{11}$$

$$d_{G_b}(\kappa, \mathscr{B}) = \inf\left\{d_{G_b}(\kappa, \mu), \mu \in \mathscr{B}\right\}, \tag{12}$$

$$d_{G_b}(\mathscr{A}, \mathscr{B}) = \inf\left\{d_{G_b}(a, b), a \in \mathscr{A}, b \in \mathscr{B}\right\}, \tag{13}$$

$$d_{G_b}(\kappa, \mu, \mathscr{C}) = \inf\left\{G_b(\kappa, \mu, \nu), \nu \in \mathscr{C}\right\}. \tag{14}$$

Lemma 11 (see [29]). *Let $(\mathscr{W}, G_b)$ be a G_b-metric space with $s \ge 1$ and $\mathscr{A}, \mathscr{B} \in \mathscr{CB}^{G_b}(\mathscr{W})$. Then, for each $a \in \mathscr{A}$, we have*

$$G_b(a, \mathscr{B}, \mathscr{B}) \le \mathscr{H}_{G_b}(\mathscr{A}, \mathscr{B}, \mathscr{B}). \tag{15}$$

Lemma 12 (see [29]). *Let $(\mathscr{W}, G_b)$ be a G_b-metric space with $s \geq 1$. If $\mathscr{A}, \mathscr{B} \in \mathscr{CB}^{G_b}(\mathscr{W})$ and $\kappa \in \mathscr{A}$, then for each $\varepsilon > 0$, there exists $\mu \in \mathscr{B}$ such that*

$$G_b(\kappa, \mu, \mu) \leq \mathscr{H}_{G_b}(\mathscr{A}, \mathscr{B}, \mathscr{B}) + \sigma. \tag{16}$$

Lemma 13 (see [35]). *Let $(\mathscr{W}, G_b)$ be a G_b-metric space with $s \geq 1$, and suppose that $\{\kappa_i\}$, $\{\mu_i\}$ and $\{\nu_i\}$ are G_b-convergent to κ, μ, and ν, respectively. Then, we have*

$$\begin{aligned} \frac{1}{s^3} G_b(\kappa, \mu, \nu) &\leq \lim_{i \longrightarrow \infty} \inf G_b(\kappa_i, \mu_i, \nu_i) \\ &\leq \lim_{i \longrightarrow \infty} \sup G_b(\kappa_i, \mu_i, \nu_i) \\ &\leq s^3 G_b(\kappa, \mu, \nu). \end{aligned} \tag{17}$$

In particular, if $\kappa = \mu = \nu$, then we have $\lim_{i \longrightarrow \infty} G_b(\kappa_i, \mu_i, \nu_i) = 0$.

Definition 14 (see [22]). Let $\mathscr{W}$ be a nonempty set. Assume $\mathscr{Q} : \mathscr{W} \longrightarrow \mathscr{W}$ and $\mathscr{P} : \mathscr{W} \longrightarrow 2^{\mathscr{W}}$ are two mappings. If $w = \mathscr{Q}\kappa \in \mathscr{P}\kappa$ for some $\kappa \in \mathscr{W}$, then κ is a coincidence point of two mapping $(\mathscr{Q}, \mathscr{P})$. Then, the coincidence point of $\mathscr{Q}$ and $\mathscr{P}$ is w. On the other hand, the mappings $\mathscr{Q}$ and $\mathscr{P}$ are said to be weakly compatible if $\mathscr{Q}\kappa \in \mathscr{P}\kappa$ for some $\kappa \in \mathscr{W}$ consequently $\mathscr{Q}\mathscr{P}\kappa \subseteq \mathscr{P}\mathscr{Q}\kappa$.

Proposition 15 (see [22]). *Let $\mathscr{W}$ be a nonempty set. Assume $\mathscr{Q} : \mathscr{W} \longrightarrow \mathscr{W}$ and $\mathscr{P} : \mathscr{W} \longrightarrow 2^{\mathscr{W}}$ are said to be weakly compatible mappings. If $u = \mathscr{Q}\kappa \in \mathscr{P}\kappa$ is a unique coincidence of $\mathscr{Q}$ and $\mathscr{P}$, u is said to be a unique common fixed point of $\mathscr{Q}$ and $\mathscr{P}$.*

Definition 16 (see [36]). Let $(\mathscr{Q}, \mathscr{P})$ be the two self-mappings on an ordered metric space $(\mathscr{W}, d, \circ)$ with $\mathscr{Q}(\mathscr{W}) \subseteq \mathscr{P}(\mathscr{W})$. For every $\kappa_0 \in \mathscr{W}$, consider the sequence $\{\kappa_i\} \in \mathscr{W}$ defined by $\mathscr{Q}\kappa_i = \mathscr{P}\kappa_{i+1}$, $\forall\ i \in \mathbb{N}_0$. A sequence $\{\mathscr{Q}\kappa_i\}$ is a $\mathscr{Q}$-$\mathscr{P}$ sequence starting at κ_0.

Wangwe and Kumar [10] gave the following definition and theorem.

Definition 17 (see [10]). Let (X, G) be a G-metric space and let f, T be two hybrid mapping on this space for $p \geq 0$ and $k_i \geq 0$, $i = 1, 2, 3, 4$, such that $\sum_{i=1}^4 k_i = 1$. We define the following expression:

$$\mathscr{M}^p_{\mathscr{G}}(\zeta, \eta, \eta) = \begin{cases} \left[k_1(G(f\zeta, f\eta, f\eta))^p + k_2(G(f\zeta, T\zeta, T\zeta))^p + k_3(G(T\eta, f\eta, f\eta))^p + k_4\left(\dfrac{G(f\zeta, T\eta, T\eta) + G(T\zeta, f\eta, f\eta)}{2}\right)^p\right]^{1/p}, \\ p \geq 0, \zeta, \eta \in X, \\ [G(f\zeta, f\zeta, f\eta)]^{K_1}[G(f\zeta, T\zeta, T\zeta)]^{K_2}[G(T\eta, f\eta, f\eta)]^{K_3}\left[\dfrac{G(T\zeta, f\eta, f\eta) + G(f\zeta, T\eta, T\eta)}{2}\right]^{K_4}, \\ p = 0, \zeta, \eta \in X. \end{cases} \tag{18}$$

Theorem 18 (see [10]). *Let (X, G) be a G-metric space, and suppose $f, T : X \longrightarrow CB(X)$ is a p-hybrid mapping with almost altering distance $\psi \in \Psi$ satisfying the following conditions:*

(a) f and T are weakly compatible

(b) f and T satisfy CLR_f property

(c) $Tx \subseteq f(X)$

(d) $T(X)$ is a G-complete subspace of X

(e) $\mathscr{H}^p_G(T\zeta, T\eta, T\eta) \leq \psi(\mathscr{M}^p_{\mathscr{G}}(\zeta, \eta, \eta))$, for all $\zeta, \eta \in X$ and $p \geq 0$

Therefore, f and T admit a unique common fixed point in X.

3. Main Results

We commence this section by extending Definition 17 to G_b-metric space setting.

Definition 19. Let $(\mathscr{W}, G_b)$ be a G_b-metric space, $\mathscr{Q}, \mathscr{P}$ be a pair of hybrid mapping, and $s \geq 1$ with $p \geq 0$ and $z_i \geq 0$, $i = 1, 2, 3, 4$, such that $\sum_{i=1}^4 z_i = 1$. Then, we define the following expression:

$$\begin{aligned} \mathscr{N}^p_{G_b}(\kappa, \mu, \nu) = {} & z_1(G_b(\mathscr{Q}\kappa, \mathscr{Q}\mu, \mathscr{Q}\nu))^p + z_2(G_b(\mathscr{Q}\kappa, \mathscr{P}\kappa, \mathscr{P}\kappa))^p \\ & + z_3(G_b(\mathscr{P}\mu, \mathscr{Q}\mu, \mathscr{Q}\nu))^p \\ & + z_4\left(\frac{G_b(\mathscr{Q}\kappa, \mathscr{P}\mu, \mathscr{P}\nu) + G_b(\mathscr{P}\kappa, \mathscr{Q}\mu, \mathscr{Q}\nu)}{s}\right)^p, \end{aligned} \tag{19}$$

for $p \geq 0, \kappa, \mu, \nu \in \mathscr{W}$.

Now, we are equipped to prove the following theorem.

Theorem 20. *Let $(\mathscr{W}, G_b)$ be a complete G_b-metric space, and suppose $\mathscr{Q} : \mathscr{W} \longrightarrow \mathscr{W}$ and $\mathscr{P} : \mathscr{W} \longrightarrow \mathscr{CB}^{G_b}(\mathscr{W})$ are p-*

hybrid contraction mapping on $\mathscr{W}$ and $s \geq 1$ satisfying the following conditions:

(i) $\exists\, \mathcal{Q}\mathscr{W} \subseteq \mathscr{P}\mathscr{W}$, *since* $(\mathscr{W}, G_b)$ *is complete*

(ii) $\mathcal{Q}$ *and* $\mathscr{P}$ *are weakly compatible*

(iii) $\mathscr{P}\mathscr{W}$ *is a* G_b*-complete subspace of* $\mathscr{W}$

(iv) For i, $\{\mathcal{Q}\kappa_i\}$ *and* $\{\mathscr{P}\kappa_i\}$ *converge to a common fixed point*

(v) $\exists$ *a constant* $(\theta = z_1 + z_2 + z_4)/(1 - (z_3 + z_4)) \in [0, 1)$, $s \geq 1$, *and* $\theta > 0$ *such that* $\forall \kappa, \mu, \nu \in \mathscr{W}$; *we have*

$$s\mathscr{H}^p_{G_b}(\mathscr{P}\kappa, \mathscr{P}\mu, \mathscr{P}\nu) \leq \theta \mathscr{N}^p_{G_b}(\kappa, \mu, \nu) \tag{20}$$

Then, $\mathcal{Q}$ *and* $\mathscr{P}$ *pose a unique coincidence fixed point.*

Proof. Assume that $\mathcal{Q}\mathscr{W} \subseteq \mathscr{P}\mathscr{W}$ and $\mathscr{P}\mathscr{W}$ is a G_b-complete subspace of $\mathscr{W}$. We can construct a $\mathscr{P}$-$\mathcal{Q}$-sequence such that $\{\mathscr{P}\kappa_i\} \in \mathscr{W}$ with initial point κ_0 satisfying

$$(\mathcal{Q}\kappa_i, \mathscr{P}\kappa_{i-1}, \mathscr{P}\kappa_{i-1}), (\mathcal{Q}\kappa_{i+1}, \mathscr{P}\kappa_i, \mathscr{P}\kappa_i), (\mathcal{Q}\kappa_{i+2}, \mathscr{P}\kappa_{i+1}, \mathscr{P}\kappa_{i+1}) \cdots, \tag{21}$$

$\forall i \in \mathbb{N}_0$, such that $\{\mathscr{P}\kappa_i\}, \{Q\kappa_i\} \in \mathscr{P}(\mathscr{W})$.

Let κ_0 be an arbitrary element in $\mathscr{W}$. If $\mathcal{Q}\kappa_0 = \mathscr{P}\kappa_0$, then κ_0 is a coincidence point of $\mathcal{Q}$ and $\mathscr{P}$. Therefore, our proof is completed. Otherwise, for $\mathcal{Q}\kappa_0 \neq \mathscr{P}\kappa_0$, it implies that $\mathcal{Q}\mathscr{W} \subset \mathscr{P}\mathscr{W}$. Now, we choose $\kappa_1 \in \mathscr{W}$ such that $\mathcal{Q}\kappa_1 = \mathscr{P}\kappa_0$. Again, we can choose $\kappa_2 \in \mathscr{W}$ such that $\mathcal{Q}\kappa_2 = \mathscr{P}\kappa_1$. By repeating the above procedure and applying Definition 16, we formulate a sequence $\{\kappa_n\} \subset \mathscr{W}$, such that

$$\begin{aligned} \mathcal{Q}\kappa_i &= \mathscr{P}\kappa_{i-1}, \\ \mathcal{Q}\kappa_{i+1} &= \mathscr{P}\kappa_i. \end{aligned} \tag{22}$$

Equivalently,

$$(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}) = (\mathscr{P}\kappa_{i-1}, \mathscr{P}\kappa_i). \tag{23}$$

Using Lemmas 11 and 12, we obtain that there exists

$$G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1}) \leq s\mathscr{H}_{G_b}(\mathscr{P}\kappa_{i-1}, \mathscr{P}\kappa_i, \mathscr{P}\kappa_i). \tag{24}$$

Consequently, we have

$$\begin{aligned} G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1}) &\leq s\mathscr{H}_{G_b}(\mathscr{P}\kappa_{i-1}, \mathscr{P}\kappa_i, \mathscr{P}\kappa_i), \\ &< \mathscr{H}_{G_b}(\mathscr{P}\kappa_{i-1}, \mathscr{P}\kappa_i, \mathscr{P}\kappa_i). \end{aligned} \tag{25}$$

Apply $\kappa = \kappa_{i-1}$, $\mu = \kappa_i$, and $\nu = \kappa_i$ in (20); we get

$$\begin{aligned} G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1})^p &\leq s\mathscr{H}^p_{G_b}(\mathscr{P}\kappa_{i-1}, \mathscr{P}\kappa_i, \mathscr{P}\kappa_i), \\ &\leq \theta \mathscr{N}^p_{G_b}(\kappa_{i-1}, \kappa_i, \kappa_i), \end{aligned} \tag{26}$$

where

$$\begin{aligned} &\mathscr{N}^p_{G_b}(\kappa_{i-1}, \kappa_i, \kappa_i) \\ &\quad = z_1(G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i))^p \\ &\qquad + z_2(G_b(\mathcal{Q}\kappa_{i-1}, \mathscr{P}\kappa_{i-1}, \mathscr{P}\kappa_{i-1}))^p \\ &\qquad + z_3(G_b(\mathscr{P}\kappa_i, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i))^p \\ &\qquad + z_4\left(\frac{G_b(\mathcal{Q}\kappa_{i-1}, \mathscr{P}\kappa_i, \mathscr{P}\kappa_i) + G(\mathscr{P}\kappa_{i-1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i)}{s}\right)^p, \end{aligned} \tag{27}$$

$$\begin{aligned} &\mathscr{N}^p_{G_b}(\kappa_{i-1}, \kappa_i, \kappa_i) \\ &\quad = z_1(G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i))^p + z_2(G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i))^p \\ &\qquad + z_3(G_b(\mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i))^p \\ &\qquad + z_4\left(\frac{G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1}) + G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i)}{s}\right)^p. \end{aligned} \tag{28}$$

By (G_b5), we have

$$\begin{aligned} &G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1}) \\ &\quad \leq s[G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i) + G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1})]. \end{aligned} \tag{29}$$

Applying (29) in (28), we obtain

$$\begin{aligned} &\mathscr{N}^p_{G_b}(\kappa_{i-1}, \kappa_i, \kappa_i) \\ &\quad = z_1(G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i))^p + z_2(G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i))^p \\ &\qquad + z_3(G_b(\mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i))^p \\ &\qquad + z_4\left(\frac{s[G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i) + G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1})] + G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i)}{s}\right)^p, \end{aligned} \tag{30}$$

$$\begin{aligned} \mathscr{N}^p_{G_b}(\kappa_{i-1}, \kappa_i, \kappa_i) = {} & (z_1 + z_2 + z_4)(G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i))^p \\ & + (z_3 + z_4)(G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1}))^p. \end{aligned} \tag{31}$$

Using (9) and (31) in (26), we get

$$\begin{aligned} &(G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1}))^p \\ &\quad < (z_1 + z_2 + z_4)(G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i))^p \\ &\qquad + (z_3 + z_4)(G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1}))^p . (1 - (z_3 + z_4)) \\ &\qquad \cdot (G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1}))^p \\ &\quad \leq (z_1 + z_2 + z_4)(G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i))^p \\ &\qquad \cdot (G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1}))^p \\ &\quad \leq \frac{z_1 + z_2 + z_4}{1 - (z_3 + z_4)}(G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i))^p. \end{aligned} \tag{32}$$

Let $\theta = (z_1 + z_2 + z_4)/(1 - (z_3 + z_4) < 1)$; we have

$$(G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1}))^p \leq \theta(G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i))^p. \tag{33}$$

By repeating the above procedure, we construct a sequence $\mathcal{Q}\kappa_i \subset \mathcal{W}$ such that $\mathcal{Q}\kappa_i \notin \mathcal{P}\kappa_i, \mathcal{Q}\kappa_{i+1} \in \mathcal{P}\kappa_i$.

$$(G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1}))^p \leq \theta^i (G_b(\mathcal{Q}\kappa_{i-1}, \mathcal{Q}\kappa_i, \mathcal{Q}\kappa_i))^p. \quad (34)$$

In order to simplify the above equation, let $G_{b_i} = (G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1}))^p$. Thus, by (34), we have

$$\sum_{i=0}^{\infty} G_{b_i} \leq \sum_{i=1}^{\infty} \theta^i G_{b_{i-1}}, \quad (35)$$

for all $i \in \mathbb{N}$.

By taking limits as $i \longrightarrow \infty$ in (35), we obtain

$$\lim_{i \longrightarrow \infty} G_{b_i} = 0. \quad (36)$$

Therefore, $\sum_{i=0}^{\infty} G_{b_i} = \sum_{i=0}^{\infty} G_{b_{i-1}}$ converges.

Using (34) and (G_b4)-symmetric properties, for all $i, j \in \mathbb{N}_0$ with $j > i$, we obtain

$$\begin{aligned} G_b\left(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+j}, \mathcal{Q}\kappa_{i+j}\right)^p \leq{}& sG_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+1})^p \\ &+ s^2 G_b(\mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+2}, \mathcal{Q}\kappa_{i+2})^p \\ &+ s^3 G_b(\mathcal{Q}\kappa_{i+1}, \mathcal{Q}\kappa_{i+2}, \mathcal{Q}\kappa_{i+2})^p \\ &+ \cdots + s^{j-2} G_b\left(\mathcal{Q}\kappa_{i+j-2}, \mathcal{Q}\kappa_{i+j-1}, \mathcal{Q}\kappa_{i+j-1}\right)^p \\ &+ s^{j-1} G_b\left(\mathcal{Q}\kappa_{i+j-1}, \mathcal{Q}\kappa_{i+j}, \mathcal{Q}\kappa_{i+j}\right)^p. \end{aligned} \quad (37)$$

On the other hand, using (35), we obtain

$$\begin{aligned} G_b\left(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+j}, \mathcal{Q}\kappa_{i+j}\right)^p &\leq sG_{b_i} + s^2 G_{b_{i+1}} \\ &\quad + s^3 G_{b_{i+2}} \cdots + s^{j-2} G_{b_{i+j-1}} + s^{j-1} G_{b_{i+j-2}}, \\ &\leq s^i \sum_{i=1}^{\infty} \theta^i G_{b_{i-1}}, \longrightarrow 0 \quad \text{as } i \longrightarrow \infty. \end{aligned} \quad (38)$$

where $G_{b_i} = G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa_{i+j}, \mathcal{Q}\kappa_{i+j})^p = 0$. This proves that the sequence $\{\mathcal{Q}\kappa_i\}$ satisfies the G_b-Cauchy sequence conditions on complete subspace $\mathcal{Q}\mathcal{W}$. Henceforth, $\{\mathcal{Q}\kappa_i\}$ is a Cauchy sequence.

Let $(\mathcal{Q}, \mathcal{P})$ be closed and weakly compatible mappings. From Definition 14, we have

$$\begin{aligned} \mathcal{Q}\mathcal{P}\kappa_i &= \mathcal{P}\mathcal{Q}\kappa_i, \\ \mathcal{Q}\kappa_i &= \mathcal{P}\kappa_i, \\ \mathcal{P}\mathcal{Q}\kappa_i &= \mathcal{P}\mathcal{P}\kappa_i = \mathcal{P}\kappa_{i+1}, \\ \mathcal{Q}\mathcal{P}\kappa_i &= \mathcal{Q}\mathcal{Q}\kappa_i = \mathcal{Q}\kappa_{i+1}. \end{aligned} \quad (39)$$

Now, we find that $\kappa \in \mathcal{W}$ such that $\mathcal{Q}\kappa = \mu$. We will show that $\mathcal{Q}\kappa \in \mathcal{P}\kappa$. For each $i \in \mathbb{N}$, using (20), it follows that

$$G_b(\mathcal{Q}\kappa_{i+1}, \mathcal{P}\kappa, \mathcal{P}\kappa)^p \leq s\mathcal{H}^p_{G_b}(\mathcal{P}\kappa_i, \mathcal{P}\kappa, \mathcal{P}\kappa) \leq \theta \mathcal{N}^p_{G_b}(\kappa_i, \kappa, \kappa), \quad (40)$$

where

$$\begin{aligned} \mathcal{N}^p_{G_b}(\kappa_i, \kappa, \kappa) = {}& z_1(G_b(\mathcal{Q}\kappa_i, \mathcal{Q}\kappa, \mathcal{Q}\kappa))^p \\ &+ z_2(G_b(\mathcal{Q}\kappa_i, \mathcal{P}\kappa_i, \mathcal{P}\kappa_i))^p \\ &+ z_3(G_b(\mathcal{P}\kappa, \mathcal{Q}\kappa, \mathcal{Q}\kappa))^p \\ &+ z_4\left(\frac{G_b(\mathcal{Q}\kappa_i, \mathcal{P}\kappa, \mathcal{P}\kappa) + G(\mathcal{P}\kappa_i, \mathcal{Q}\kappa, \mathcal{Q}\kappa)}{s}\right)^p. \end{aligned} \quad (41)$$

Taking the limit as $i \longrightarrow \infty$ in (40) and (41) with $\theta < 1$, we obtain

$$\begin{aligned} &G_b(\mathcal{Q}\kappa, \mathcal{P}\kappa, \mathcal{P}\kappa)^p \\ &\quad \leq z_1(G_b(\mathcal{Q}\kappa, \mathcal{Q}\kappa, \mathcal{Q}\kappa))^p + z_2(G_b(\mathcal{Q}\kappa, \mathcal{P}\kappa, \mathcal{P}\kappa))^p \\ &\qquad + z_3(G_b(\mathcal{P}\kappa, \mathcal{Q}\kappa, \mathcal{Q}\kappa))^p \\ &\qquad + z_4\left(\frac{G_b(\mathcal{Q}\kappa, \mathcal{P}\kappa, \mathcal{P}\kappa) + G(\mathcal{P}\kappa, \mathcal{Q}\kappa, \mathcal{Q}\kappa)}{s}\right)^p, \\ &\quad \leq z_1(G_b(\mathcal{Q}\kappa, \mathcal{Q}\kappa, \mathcal{Q}\kappa))^p + z_2(G_b(\mathcal{Q}x, \mathcal{P}\kappa, \mathcal{P}\kappa))^p \\ &\qquad + z_3(G_b(\mathcal{P}\kappa, \mathcal{Q}\kappa, \mathcal{Q}\kappa))^p \\ &\qquad + \left(2s^{-1}\right)^p z_4(G_b(\mathcal{Q}\kappa, \mathcal{P}\kappa, \mathcal{P}\kappa))^p, G_b(\mathcal{Q}\kappa, \mathcal{P}\kappa, \mathcal{P}\kappa)^p \\ &\quad \leq \left(z_2 + z_3 + \left(2s^{-1}\right)^p z_4\right) G_b(\mathcal{Q}\kappa, \mathcal{P}\kappa, \mathcal{P}\kappa)^p. \\ &\left(1 - \left(z_2 + z_3 + \left(2s^{-1}\right)^p z_4\right)\right) G_b(\mathcal{Q}\kappa, \mathcal{P}\kappa, \mathcal{P}\kappa)^p \\ &\quad \leq 0, \\ &(\mathcal{Q}\kappa, \mathcal{P}\kappa, \mathcal{P}\kappa)^p \\ &\quad \leq 0. \end{aligned} \quad (42)$$

This shows that $\mathcal{Q}\kappa \in \mathcal{P}\kappa$. That is, $\mathcal{Q}$ and $\mathcal{P}$ have a point of coincidence.

Next, we prove the uniqueness of the point of coincidence of $\mathcal{Q}$ and $\mathcal{P}$. Let $\mathcal{Q}\kappa \in \mathcal{P}\kappa$ and $\mathcal{Q}\mu \in \mathcal{P}\mu$. Assume that $G_b(\mathcal{Q}\kappa, \mathcal{P}\mu, \mathcal{P}\mu)^p > 0$. Using (20), we get

$$G_b(\mathcal{Q}\kappa, \mathcal{P}\mu, \mathcal{P}\mu)^p \leq s\mathcal{H}^p_{G_b}(\mathcal{P}\kappa, \mathcal{P}\mu, \mathcal{P}\mu) \leq \theta \mathcal{N}^p_{G_b}(\kappa, \mu, \mu), \quad (43)$$

where

$$\begin{aligned} \mathcal{N}^p_{G_b}(\kappa, \mu, \mu) = {}& z_1(G_b(\mathcal{Q}\kappa, \mathcal{Q}\mu, \mathcal{Q}y))^p + z_2(G_b(\mathcal{Q}\kappa, \mathcal{P}\kappa, \mathcal{P}\kappa))^p \\ &+ z_3(G_b(\mathcal{P}\mu, \mathcal{Q}\mu, \mathcal{Q}\mu))^p \\ &+ z_4\left(\frac{G_b(\mathcal{Q}\mu, \mathcal{P}\mu, \mathcal{P}\mu) + G(\mathcal{P}\kappa, \mathcal{Q}\mu, \mathcal{Q}\mu)}{s}\right)^p. \end{aligned} \quad (44)$$

From (G_b4) the symmetric properties, using (43) and (44) with $\theta < 1$, we obtain

$$\begin{aligned}
&G_b(\mathcal{Q}\kappa, \mathcal{P}\mu, \mathcal{P}\mu)^p \\
&\quad \le z_1(G_b(\mathcal{Q}\kappa, \mathcal{Q}\mu, \mathcal{Q}\mu))^p + z_2(G_b(\mathcal{Q}\kappa, \mathcal{P}\kappa, \mathcal{P}\kappa))^p \\
&\qquad + z_3(G_b(\mathcal{P}\mu, \mathcal{Q}\mu, \mathcal{Q}\mu))^p \\
&\qquad + z_4\left(\frac{G_b(\mathcal{Q}\kappa, \mathcal{P}\mu, \mathcal{P}\mu) + G(\mathcal{P}\kappa, \mathcal{Q}\mu, \mathcal{Q}\mu)}{s}\right)^p, \\
&G_b(\mathcal{Q}\kappa, \mathcal{P}\mu, \mathcal{P}\mu)^p \\
&\quad \le \left(z_1 + \left(2s^{-1}\right)^p z_4\right) G_b(\mathcal{Q}\kappa, \mathcal{P}\mu, \mathcal{P}\mu)^p, \\
&\left(1 - \left(z_1 + \left(2s^{-1}\right)^p z_4\right)\right) G_b(\mathcal{Q}\kappa, \mathcal{P}\mu, \mathcal{P}\mu)^p \\
&\quad \le 0, \\
&(\mathcal{Q}\kappa, \mathcal{P}\mu, \mathcal{P}\mu)^p \\
&\quad \le 0.
\end{aligned} \tag{45}$$

We can conclude that $\mathcal{Q}\kappa = \mathcal{P}\mu$. Suppose that $\mathcal{Q}$ and $\mathcal{P}$ are weakly compatible. By applying Proposition 15 and Definition 14, we obtain that $\mathcal{Q}$ and $\mathcal{P}$ have a unique coincidence fixed point, which is a contradiction. Thus, μ is a coincidence point of $\mathcal{Q}$ and $\mathcal{P}$, for $\kappa = \mu$. The proof is completed. □

Inspired by the idea of Theorem 20, we can deduce the corollary as follows:

Corollary 21. *Let $(\mathcal{W}, G_b)$ be a G_b-metric space, and let $\mathcal{Q} : \mathcal{W} \longrightarrow \mathcal{W}$ and $\mathcal{P} : \mathcal{W} \longrightarrow \mathcal{CB}^{G_b}(\mathcal{W})$ be a p-hybrid mapping which satisfies the following hypotheses:*

(i) $\mathcal{Q}\mathcal{W} \subset (\mathcal{W}, G_b)$

(ii) $\exists$ a continuous function $\theta < 1$ and $s \ge 1$ such that

$$s\mathcal{H}^p_{G_b}(\mathcal{P}\kappa, \mathcal{P}\mu, \mathcal{P}\nu) \le \theta \mathcal{M}^p_{G_b}(\kappa, \mu, \nu), \tag{46}$$

where

$$\mathcal{M}^p_{G_b}(\kappa, \mu, \nu) = \left\{ \begin{array}{c} G_b(\mathcal{Q}\kappa, \mathcal{Q}\mu, \mathcal{Q}\nu), G_b(\mathcal{Q}\kappa, \mathcal{P}\kappa, \mathcal{Q}\nu) \\ G_b(\mathcal{Q}\nu, \mathcal{P}\nu, \mathcal{Q}\nu), G_b(\mathcal{Q}\mu, \mathcal{Q}\mu, \mathcal{Q}\nu) \end{array} \right\} \tag{47}$$

Then, Q and $\mathcal{P}$ have the unique coincidence fixed point.

Proof. We prove the above corollary by following similar steps of Theorem 20. Therefore, the proof is completed. □

Next, we demonstrate with an example for Theorem 20.

Example 6. Let $\mathcal{W} = [0,\infty)$ be endowed with the usual ordering on $\mathbb{R}$ and G_b-metric on $\mathcal{W}$ be given by $G_b(\kappa, \mu, \nu) = (1/9)(|\kappa - \mu| + |\mu - \nu| + |\kappa - \nu|)^2$, where $s = 2^{p-1}$ and $p \ge 1$ with $z_1 = 0.5, z_2 = 0.1,\ z_3 = 0.2$, and $z_4 = 0.1$.

Consider $\mathcal{W}$ to be G_b-complete. Define a self-map as $\mathcal{Q} : \mathcal{W} \longrightarrow \mathcal{W}$ by

$$\mathcal{Q}\kappa = \sqrt[3]{\kappa}, \quad \forall \kappa \in \mathcal{W}, \tag{48}$$

and $\mathcal{P} : \mathcal{W} \longrightarrow \mathcal{CB}^{G_b}(\mathcal{W})$ by

$$\mathcal{P}\kappa = [0, \kappa^n], \quad \forall \kappa \in \mathcal{W}. \tag{49}$$

Therefore,

(i) $\mathcal{P}\mathcal{W}$ is a G_b-complete subspace in $\mathcal{W}$

(ii) for $\theta \in [0, 1)$ in (20), we have

$$s\mathcal{H}^p_{G_b}(\mathcal{P}\kappa, \mathcal{P}\mu, \mathcal{P}\nu) \le \theta \mathcal{N}^p_{G_b}(\kappa, \mu, \nu), \tag{50}$$

where

$$\begin{aligned}
\mathcal{N}^p_{G_b}(\kappa, \mu, \nu) &= z_1(G_b(\mathcal{Q}\kappa, \mathcal{Q}\mu, \mathcal{Q}\nu))^p + z_2(G_b(\mathcal{Q}\kappa, \mathcal{P}\kappa, \mathcal{P}\kappa))^p \\
&\quad + z_3(G_b(\mathcal{P}\mu, \mathcal{Q}\mu, \mathcal{Q}\nu))^p \\
&\quad + z_4\left(\frac{G_b(\mathcal{Q}\kappa, \mathcal{P}\mu, \mathcal{P}\nu) + G_b(\mathcal{P}\kappa, \mathcal{Q}\mu, \mathcal{Q}\nu)}{s}\right)^p,
\end{aligned} \tag{51}$$

for all $\kappa, \mu, \nu \in \mathcal{W}$

Applying (i), we prove that $\mathcal{P}\mathcal{W}$ is a G_b-complete subspace in $\mathcal{W}$. By Proposition 7, we have

$$G_b(\kappa, \mu, \mu) \le \frac{1}{9}(|\kappa - \mu| + |\mu - \mu| + |\kappa - \mu|)^2 = \frac{4}{9}|\kappa - \mu|^2, \tag{52}$$

$$G_b(\mu, \kappa, \kappa) \le \frac{1}{9}(|\mu - \kappa| + |\kappa - \kappa| + |\mu - \kappa|)^2 = \frac{4}{9}|\mu - \kappa|^2. \tag{53}$$

By (52) and ((53)) in (9), we obtain

$$d_{G_b}(\kappa, \mu) \le \frac{4}{9}|\kappa - \mu|^2 + \frac{4}{9}|\mu - \kappa|^2 = \frac{8}{9}|\kappa - \mu|^2. \tag{54}$$

From (ii), assume that $\kappa, \mu, \nu \in \mathcal{W}$. If $\kappa = \mu = \nu = 0$, then $\mathcal{P}\kappa = \mathcal{P}\mu = \mathcal{P}\nu = 0$ and $s\mathcal{H}_{G_b}(\mathcal{P}\kappa, \mathcal{P}\mu, \mathcal{P}\nu) = 0$. The proof is completed. Otherwise, we suppose that the value of κ, μ, ν are not all zero.

For $\kappa \le \mu \le \nu$, we get

$$\mathcal{H}^p_{G_b}(\mathcal{P}\kappa, \mathcal{P}\mu, \mathcal{P}\nu) = \mathcal{H}^p_{G_b}([0, \kappa^n], [0, \mu^n], [0, \nu^n]). \tag{55}$$

By (10), (55) is equivalent to

$$= \max \begin{Bmatrix} \sup\limits_{0\le a\le \kappa^n} G_b(a, [0, \mu^n], [0, \nu^n])^p \\ \sup\limits_{0\le b\le \mu^n} G_b(b, [0, \kappa^n], [0, \nu^n])^p \\ \sup\limits_{0\le c\le \nu^n} G_b(c, [0, \kappa^n], [0, \mu^n])^p \end{Bmatrix}. \tag{56}$$

Since $\kappa \le \mu \le \nu$, then $[0, \kappa^n] \subseteq [0, \mu^n] \subseteq [0, \nu^n]$, using (11) yields

$$\begin{aligned} &d_{G_b}([0, \kappa^n], [0, \mu^n])^p = 0, \\ &d_{G_b}([0, \mu^n], [0, \nu^n])^p = 0, \\ &d_{G_b}([0, \kappa^n], [0, \nu^n])^p = 0. \end{aligned} \tag{57}$$

Now, for each $0 \le a \le \kappa^n$ and $d_{G_b}(\kappa, \mu) = (8/9)|\kappa - \mu|^2$ in (11) and (12), we have

$$\begin{aligned} &G_b(a, [0, \mu^n], [0, \nu^n])^p \\ &\quad = \left(d_{G_b}(a, [0, \mu^n]) + d_{G_b}([0, \mu^n], [0, \mu^n]) + d_{G_b}(a, [0, \nu^n])\right)^p \\ &\quad \le \left(\frac{8}{9}|a - \mu^n|^2 + 0 + \frac{8}{9}|a - \nu^n|^2\right)^p \\ &\quad = \left(\frac{8}{9}|a - \mu^n|^2 + \frac{8}{9}|a - \nu^n|^2\right)^p \\ &\quad = \left(\frac{8}{9}\right)^p \left(|\kappa^n - \mu^n|^2 + \frac{8}{9}|\kappa^n - \nu^n|^2\right)^p. \end{aligned} \tag{58}$$

Next, for each $0 \le b \le \mu^n$ and $d_{G_b}(\kappa, \mu) = (8/9)|\kappa - \mu|^2$ in (11) and (12), we get

$$\begin{aligned} &G_b(b, [0, \kappa^n], [0, \nu^n])^p \\ &\quad = \left(d_{G_b}(b, [0, \kappa^n]) + d_{G_b}([0, \kappa^n], [0, \nu^n]) + d_{G_b}(b, [0, \nu^n])\right)^p \\ &\quad \le \left(\frac{8}{9}|b - \kappa^n|^2 + 0 + \frac{8}{9}|b - \nu^n|^2\right)^p \\ &\quad = \left(\frac{8}{9}|b - \kappa^n|^2 + \frac{8}{9}|b - \nu^n|^2\right)^p \\ &\quad = \left(\frac{8}{9}\right)^p \left(|\mu^n - \kappa^n|^2 + |\kappa^n - \nu^n|^2\right)^p. \end{aligned} \tag{59}$$

Using the same approach, for each $0 \le \nu^n$, ((11)) and ((12)) give

$$\begin{aligned} &G_b(c, [0, \kappa^n], [0, \mu^n])^p \\ &\quad = \left(d_{G_b}(c, [0, \kappa^n]) + d_{G_b}([\kappa^n], [0, \mu^n]) + d_{G_b}(c, [0, \mu^n])\right)^p \\ &\quad \le \left(\frac{8}{9}|c - \kappa^n|^2 + 0 + \frac{8}{9}|c - \mu^n|^2\right)^p \\ &\quad = \left(\frac{8}{9}|c - \kappa^n|^2 + \frac{8}{9}|c - \mu^n|^2\right)^p \\ &\quad = \left(\frac{8}{9}\right)^p \left(|\nu^n - \kappa^n|^2 + |\nu^n - \mu^n|^2\right)^p. \end{aligned} \tag{60}$$

Consequently, using (58), (59), and (60) in (55), we obtain

$$\begin{aligned} &\mathscr{H}^p_{G_b}(\mathscr{P}\kappa, \mathscr{P}\mu, \mathscr{P}\nu) \\ &\quad \le \max \begin{Bmatrix} \sup\limits_{0\le a\le \kappa^n} \left(\frac{8}{9}\right)^p \left(|\kappa^n - \mu^n|^2 + |\kappa^n - \mu^n|^2\right)^p \\ \sup\limits_{0\le b\le \mu^n} \left(\frac{8}{9}\right)^p \left(|\mu^n - \kappa^n|^2 + |\kappa^n - \nu^n|^2\right)^p \\ \sup\limits_{0\le c\le \nu^n} \left(\frac{8}{9}\right)^p \left(|\nu^n - \kappa^n|^2 + |\nu^n - \mu^n|^2\right)^p \end{Bmatrix} \\ &\quad = \left(\frac{8}{9}\right)^p \left(|\nu^n - \kappa^n|^2 + |\nu^n - \mu^n|^2\right)^p. \end{aligned} \tag{61}$$

Further, we calculate the following G_b-metrics. Equations ((11)) and ((12)) yield

$$\begin{aligned} G_b(\mathscr{Q}\kappa, \mathscr{Q}\mu, \mathscr{Q}\nu)^p &= G_b\left(\sqrt[3]{\kappa}, \sqrt[3]{\mu}, \sqrt[3]{\nu}\right)^p \\ &= \left(\frac{1}{9}\right)^p \left(\left|\sqrt[3]{\kappa} - \sqrt[3]{\mu}\right| + \left|\sqrt[3]{\mu} - \sqrt[3]{\nu}\right| \right. \\ &\quad \left. + \left|\sqrt[3]{\kappa} - \sqrt[3]{\nu}\right|\right)^{2p}. \end{aligned} \tag{62}$$

In the same manner, we can calculate the following G_b-metrics.

$$\begin{aligned} G_b(\mathscr{Q}\kappa, \mathscr{P}\kappa, \mathscr{P}\kappa)^p &= G_b\left(\sqrt[3]{\kappa}, [0, \kappa^n], [0, \kappa^n]\right)^p \\ &= \left(\frac{16}{9}\right)^p \left|\sqrt[3]{\kappa} - \kappa^n\right|^{2p}, \\ G_b(\mathscr{P}\mu, \mathscr{Q}\mu, \mathscr{Q}\nu) &= G_b\left([0, \mu^n], \sqrt[3]{\mu}, \sqrt[3]{\nu}\right)^p \\ &= \left(\frac{8}{9}\right)^p \left(\left|\mu^n - \sqrt[3]{\mu}\right|^2 + \left|\sqrt[3]{\mu} - \sqrt[3]{\nu}\right|^2 \right. \\ &\quad \left. + \left|\mu^n - \sqrt[3]{\nu}\right|^2\right)^p, \\ G_b(\mathscr{Q}\kappa, \mathscr{P}\mu, \mathscr{P}\nu)^p &= G_b\left(\sqrt[3]{\kappa}, [0, \mu^n], [0, \nu^n]\right)^p \\ &= \left(\frac{8}{9}\right)^p \left(\left|\sqrt[3]{\kappa} - \mu^n\right|^2 + \left|\sqrt[3]{\kappa} - \nu^n\right|^2\right)^p, \\ G_b(\mathscr{P}\kappa, \mathscr{Q}\mu, \mathscr{Q}\nu)^p &= G_b\left([0, \kappa^n], \sqrt[3]{\mu}, \sqrt[3]{\nu}\right)^p \\ &= \left(\frac{8}{9}\right)^p \left(\left|\kappa^n - \sqrt[3]{\mu}\right|^2 + \left|\sqrt[3]{\mu} - \sqrt[3]{\nu}\right|^2 \right. \\ &\quad \left. + \left|\kappa^n - \sqrt[3]{\nu}\right|^2\right)^p. \end{aligned} \tag{63}$$

Applying the above equality in (20), we obtain

$$
\begin{aligned}
&s\left(\frac{8}{9}\right)^{p}\left(|\nu^{n}-\kappa^{n}|^{2}+|\nu^{n}-\mu^{n}|^{2}\right)^{p}\\
&\leq\theta\left\{\begin{array}{c}
z_{1}\left(\frac{1}{9}\right)^{p}\left(\left|\sqrt[3]{\kappa}-\sqrt[3]{\mu}\right|+\left|\sqrt[3]{\mu}-\sqrt[3]{\nu}\right|+\left|\sqrt[3]{\kappa}-\sqrt[3]{\nu}\right|\right)^{2p}+z_{2}\left(\frac{16}{9}\right)^{p}\left|\sqrt[3]{\kappa}-\kappa^{n}\right|^{2p}+z_{3}\left(\frac{8}{9}\right)^{p}\left(\left|\mu^{n}-\sqrt[3]{\mu}\right|^{2}+\left|\sqrt[3]{\mu}-\sqrt[3]{\nu}\right|^{2}+\left|\mu^{n}-\sqrt[3]{\nu}\right|^{2}\right)^{p}\\
\left(\frac{8}{9}\right)^{p}\left(\left|\sqrt[3]{\kappa}-\mu^{n}\right|^{2}+\left|\sqrt[3]{\kappa}-\nu^{n}\right|^{2}\right)^{p}+\left(\frac{8}{9}\right)^{p}\left(\left|\kappa^{n}-\sqrt[3]{\mu}\right|^{2}+\left|\sqrt[3]{\mu}-\sqrt[3]{\nu}\right|^{2}++z_{4}\left(\frac{\left(\left|\kappa^{n}-\sqrt[3]{\nu}\right|^{2}\right)^{p}}{s}\right)\right)
\end{array}\right\}.
\end{aligned}
\tag{64}
$$

By using the above inequality, for $G_b(1/4, 1/3, 1/2), n = 1, p = 2, s = 2^{p-1} = s = 2,$ and $z_1 = 0.5, z_2 = 0.1, z_3 = 0.2,$ and $z_4 = 0.1$, it follows that

$$
\begin{aligned}
2\times 0.006439567 &\leq \theta\{0.5\times 0.000160182+0.1\\
&\quad \times 0.065873215+0.2\times 0.097691847\\
&\quad +0.1\times 0.020276367\},\\
0.012879134 &\leq \theta\{0.000080091+0.006587321\\
&\quad +0.019538369+0,020276367\}\\
0.012879134 &\leq 0.0464821148\theta,
\end{aligned}
\tag{65}
$$

for $\theta = (z_1 + z_2 + z_4)/(1 - (z_3 + z_4) \leq 1)$.

This shows that all $\kappa, \mu, \nu \in \mathscr{W}$ and the hypothesis given in Theorem 20 are correct. Therefore, p-hybrid mappings have an amazing coincidence fixed point. So, $\kappa = 0$ is a unique coincidence fixed point of $\mathscr{Q}$ and $\mathscr{P}$. Clearly, $\mathscr{W}$ is $(\mathscr{P}, \mathscr{Q})$-closed, and $\kappa_0 = 0, (\mathscr{Q}0, \mathscr{P}0, \mathscr{P}0) \in \mathscr{W}$.

4. An Application to Nonlinear Fractional Boundary Valued Problem in G_b-Metric Space

The nonlinear fractional differential equation is used as convolution mapping. Convolution and associated functions are found in many sciences, engineering, and mathematics applications, such as the following:

(i) In physics: the system linear system with a "superposition principle," a convolution operation, makes an appearance

(ii) Used in acoustics: Doppler effect of the sound which is a convolution

(iii) In image processing: in digital image processing, convolutional filtering plays a vital role in many essential algorithms in edge detection and related processes

(iv) In optics: an out-of-focus photograph is a convolution of the sharp image with a lens function

(v) In radiotherapy: Most parts of all modern codes of calculation apply the convolution-superposition algorithm in the treatment of planning systems

(vi) In electric transmission lines

(vii) In a control system: to increase the speed of response, decrease the relative stability, and decrease or eliminate the steady-state error

(viii) Application of fractional-order circuit models for modelling human tissue, plant physiology, and respiratory system

(ix) Used in the tautochrone problem: a cycloid

For further literature, we refer the reader to [37–39] and the references contained.

The Caputo derivative is used for modelling phenomena that account for interaction within the past and problems with nonlocal properties. In this case, one can think of the equation as having memory; the groundwater equation within confined, unconfined, leaky aquifers; and other diffusion problems. In addition, fractional differential arises in a dynamic process, rheology, fluid flows, viscoelasticity, chemical physics, electrical network, and numerous other shorts of science and engineering.

Inspired by [40, 41], we demonstrate the result of Theorem 20 using a nonlinear fractional boundary valued problem by transforming it into a system of integral equations.

Now, we investigate the Caputo derivative with the fractional order of the nonlinear fractional differential equation.

Definition 22 (see [41]). For continuous function f : $[0,\infty) \longrightarrow \mathbb{R}$, the Caputo derivative of functional order q is defined as

$$
{}^{C}D_{t}^{q}f(t)=\frac{1}{\Gamma(n-q)}\int_{0}^{t}(t-s)^{n-q-1}f^{n}(s)ds,\ (n-1<q<n, n=[q]+1),
\tag{66}
$$

where $[q]$ denotes the integer part of the real number q.

Definition 23 (see [41]). The Riemann-Liouville of fractional-order q for continuous function $f(t)$ is defined as

$$D^q f(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s)ds, (n = [q]+1), \tag{67}$$

provided that the right-hand side is point-wise defined on $(0, \infty)$.

Definition 24 (see [41]). The Riemann-Liouville of the fractional integral of order q for continuous function $f(t)$ is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \quad q > 0, \tag{68}$$

provided that such integral exists.

The Caputo fractional differential equation and the nonlinear fractional equation are used as convolution mapping, which has several applications in science, engineering, and mathematics, i.e., optics and radiotherapy.

We consider the following Caputo nonlinear boundary value fractional differential equation, defined as follows:

$$\begin{cases} {}^C D^q \kappa(t) = f(t, \kappa(t)), & t \in (0,1), 1 < \alpha \le 2, \\ \kappa(0) = 0, \alpha[I^p \kappa](\eta) = \kappa(1), \end{cases} \tag{69}$$

where ${}^C D^q$ denotes the Caputo fractional derivative of order q and $f : [0,1] \longrightarrow \mathscr{W}$ is a continuous function and $\alpha \ne (\Gamma(p+2))/\eta^{p+1}$, $1 < q \le 2$.

Let $\mathscr{W} = C([0,1])$ be the set of real continuous functions defined on $[0,1]$, for $\kappa, \mu, \nu \in \mathscr{W}$, and define the G_b-metric $G_b : \mathscr{W} \times \mathscr{W} \times \mathscr{W} \longrightarrow \mathbb{R}_+$ given by

$$G_b(\kappa, \mu, \nu) = \frac{1}{9}\left[\sup_{t\in[0,1]} |\kappa - \mu| + \sup_{t\in[0,1]} |\mu - \nu| + \sup_{t\in[0,1]} |\kappa - \nu|\right]^2, \tag{70}$$

$\forall \kappa, \mu, \nu \in \mathscr{W}$. Then, $(\mathscr{W}, G_b)$ is a complete G_b-metric space with $s = 2$. For $\mu = \nu$, the above inequality becomes

$$G_b(\kappa, \mu, \mu) = \frac{4}{9}\left[\sup_{t\in[0,1]} |\kappa - \mu|\right]^2. \tag{71}$$

The nonlinear fractional differential equation (69) can be transformed to integral equation as follows:

$$\begin{aligned} \kappa(t) = {} & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \kappa(s))ds \\ & - \frac{\Gamma(p+2)t}{\Gamma(q)(\Gamma(p+2) - \alpha\eta^{p+1})} \int_0^1 (1-s)^{\alpha-1} f(s, \kappa(s))ds \\ & + \frac{\alpha p(p+1)t}{\Gamma(q)(\Gamma(p+2) - \alpha\eta^{p+1})} \\ & \cdot \int_0^\eta \left[\int_0^s (\eta - s)^{p-1}(s-z)^{q-1} f(z, \kappa(z))dz\right] ds. \end{aligned} \tag{72}$$

Now, we investigate the theorem below.

Theorem 25. *Consider the hypotheses below:*

(i) $f \in C(I \times \mathscr{W}, \mathscr{W})$ *is continuous*

(ii) $\exists$ *an increasing function* $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}_+$, *such that*

$$|f(t, \kappa(s)) - f(t, \mu(s)| \le \frac{\theta}{s} \mathscr{N}^p_{G_b}(\kappa, \mu, \nu), \tag{73}$$

where $\mathscr{N}^p_{G_b}(\kappa, \mu, \nu) = (4/9)|\kappa(s) - \mu(s)|$ *and*

$$\begin{aligned} \mathscr{N}^p_{G_b}(\kappa, \mu, \nu) &= z_1(G_b(\mathscr{Q}\kappa, \mathscr{Q}\mu, \mathscr{Q}\nu))^p + z_2(G_b(\mathscr{Q}\kappa, \mathscr{P}\kappa, \mathscr{P}\kappa))^p \\ &\quad + z_3(G_b(\mathscr{P}\mu, \mathscr{Q}\mu, \mathscr{Q}\nu))^p \\ &\quad + z_4\left(\frac{G_b(\mathscr{Q}\kappa, \mathscr{P}\mu, \mathscr{P}\nu) + G_b(\mathscr{P}\kappa, \mathscr{Q}\mu, \mathscr{Q}\nu)}{s}\right)^p, \end{aligned} \tag{74}$$

for $p \ge 0, \kappa, \mu, \nu \in \mathscr{W}$

(iii) There exists $(\theta/s) \in [0, 1)$ *such that*

$$\begin{aligned} & \left[\frac{t^q}{q\Gamma(q)} + \frac{\Gamma(p+2)t}{q\Gamma(q)(\Gamma(p+2) - \alpha\eta^{p+1})}\right. \\ & \left. + \frac{\alpha p(p+1)t}{\Gamma(q)(\Gamma(p+2) - \alpha\eta^{p+1})}\right]^2 \le \frac{\theta}{s} \end{aligned} \tag{75}$$

Then, equation (69) has a coincidence solution as a fixed point $\kappa \in C(I, \mathscr{W})$.

Proof. Let us define $\mathscr{P}\mathscr{Q} : C([0,1]) \longrightarrow C([0,1])$ by

$$\begin{aligned}\mathscr{P}\mathscr{Q}\kappa(t) = {} & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,\kappa(s))ds \\ & - \frac{\Gamma(p+2)t}{\Gamma(q)(\Gamma(p+2)-\alpha\eta^{p+1})}\int_0^1 (1-s)^{q-1} f(s,\kappa(s))ds \\ & + \frac{\alpha p(p+1)t}{\Gamma(q)(\Gamma(p+2)-\alpha\eta^{p+1})} \\ & \cdot \int_0^\eta \left[\int_0^s (\eta-s)^{p-1}(s-z)^{q-1} f(z,\kappa(z))dz\right]ds,\end{aligned} \tag{76}$$

for $t \in [0,1]$; then, $\mathscr{P}\mathscr{Q}$ is a compatible and continuous mapping, such that

$$\mathscr{Q}\kappa(t) = \int_0^q (s-z)^{\alpha-1} f(z,\kappa(z))dz. \tag{77}$$

This implies that $\mathscr{Q} \in \mathscr{P}\mathscr{Q}$ and $\mathscr{Q}$ possess a coincidence fixed point $\kappa^* \in \mathscr{P}\mathscr{Q}$. To prove the existence of the coincidence point of $\mathscr{P}\mathscr{Q}$, we prove that $\mathscr{P}\mathscr{Q}$ is continuous and a contraction. To see that $\mathscr{P}\mathscr{Q}$ is continuous, assume that $\mathscr{P}\mathscr{Q}\kappa \neq \mathscr{P}\mathscr{Q}\mu$, for all $\kappa, \mu \in [0,1]$.

For $\kappa, \mu, \nu \in C([0,1])$, $\mu = \nu$ with $\kappa < \mu$, we claim that $\mathscr{H}^p_{G_b}(\mathscr{P}\kappa, \mathscr{P}\mu, \mathscr{P}\mu) \nleq (\theta/s)\mathscr{N}^p_{G_b}(\kappa,\mu,\mu)$.

By hypothesis (ii), we have

$$\begin{aligned}& \mathscr{H}^p_{G_b}(\mathscr{P}\kappa, \mathscr{P}\mu, \mathscr{P}\mu) \\ & \quad \leq \frac{4}{9}\left[\sup_{t\in[0,1]}|\kappa-\mu|\right]^2, \\ & |\mathscr{P}\mathscr{Q}\kappa - \mathscr{P}\mathscr{Q}\mu| \\ & \quad = \left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,\kappa(s))ds - \frac{\Gamma(p+2)t}{\Gamma(q)(\Gamma(p+2)-\alpha\eta^{p+1})}\right. \\ & \qquad \cdot \int_0^1 (1-s)^{q-1} f(s,\kappa(s))ds + \frac{\alpha p(p+1)t}{\Gamma(q)(\Gamma(p+2)-\alpha\eta^{p+1})} \\ & \qquad \cdot \int_0^\eta \left[\int_0^s (\eta-s)^{p-1}(s-z)^{q-1} f(z,\kappa(z))dz\right]ds \\ & \qquad - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,\mu(s))ds \\ & \qquad + \frac{\Gamma(p+2)t}{\Gamma(q)(\Gamma(p+2)-\alpha\eta^{p+1})}\int_0^1 (1-s)^{q-1} f(s,\mu(s))ds \\ & \qquad - \frac{\alpha p(p+1)t}{\Gamma(q)(\Gamma(p+2)-\alpha\eta^{p+1})} \\ & \qquad \left.\cdot \int_0^\eta \left[\int_0^s (\eta-s)^{p-1}(s-z)^{q-1} f(z,\mu(z))dz\right]ds\right]^2 \\ & \quad \leq \left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} |f(s,\kappa(s)) - f(s,\mu(s))|ds\right. \\ & \qquad + \frac{\Gamma(p+2)t}{\Gamma(q)(\Gamma(p+2)-\alpha\eta^{p+1})} \\ & \qquad \cdot \int_0^1 (1-s)^{q-1} |f(s,\kappa(s)) - f(s,\mu(s))|ds, \\ & \qquad + \frac{\alpha p(p+1)t}{\Gamma(q)(\Gamma(p+2)-\alpha\eta^{p+1})} \\ & \qquad \left.\cdot \int_0^\eta \left[\int_0^s (\eta-s)^{p-1}(s-z)^{q-1} |f(z,\kappa(z)) - f(z,\mu(z))|dz\right]ds\right]^2 \\ & \quad \leq \frac{4}{9}\left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|\kappa(s)-\mu(s)|ds\right. \\ & \qquad + \frac{\Gamma(p+2)t}{\Gamma(q)(\Gamma(p+2)-\alpha\eta^{p+1})}\int_0^1 (1-s)^{q-1}|\kappa(s)-\mu(s)|ds \\ & \qquad + \frac{\alpha p(p+1)t}{\Gamma(q)(\Gamma(p+2)-\alpha\eta^{p+1})} \\ & \qquad \left.\cdot \int_0^\eta \left[\int_0^s (\eta-s)^{p-1}(s-z)^{q-1}|\kappa(z)-\mu(z)|dz\right]ds\right]^2 \\ & \quad = \frac{4}{9}\left[\frac{t^q}{q\Gamma(q)}\|\kappa(s)-\mu(s)\|_\infty\right. \\ & \qquad + \frac{\Gamma(p+2)t}{q\Gamma(q)(\Gamma(p+2)-\alpha\eta^{p+1})}\|\kappa(s)-\mu(s)\|_\infty \\ & \qquad \left. + \frac{\alpha p(p+1)t}{\Gamma(q)(\Gamma(p+2)-\alpha\eta^{p+1})}\|\kappa(s)-\mu(s)\|_\infty\right]^2 \\ & \quad \leq \left[\frac{t^q}{q\Gamma(q)} + \frac{\Gamma(p+2)t}{q\Gamma(q)(\Gamma(p+2)-\alpha\eta^{p+1})}\right. \\ & \qquad \left. + \frac{\alpha p(p+1)t}{\Gamma(q)(\Gamma(p+2)-\alpha\eta^{p+1})}\right]^2 \frac{4}{9}\|\kappa(s)-\mu(s)\|^2_\infty \\ & \quad \leq \frac{4\theta}{9s}\|\kappa(s)-\mu(s)\|^2_\infty = \frac{4\theta}{9s}\left[\sup_{t\in[0,1]}|\kappa-\mu|\right]^2 = \frac{\theta}{s}\mathscr{N}^p_{G_b}(\kappa,\mu,\nu).\end{aligned} \tag{78}$$

This implies that

$$\mathscr{H}^p_{G_b}(\mathscr{P}\kappa, \mathscr{P}\mu, \mathscr{P}\mu) = |\mathscr{P}\mathscr{Q}\kappa - \mathscr{P}\mathscr{Q}\mu| \leq \frac{\theta}{s}\mathscr{N}^p_{G_b}(\kappa,\mu,\nu), \tag{79}$$

equivalent to

$$s\mathscr{H}^p_{G_b}(\mathscr{P}\kappa, \mathscr{P}\mu, \mathscr{P}\mu) \leq \theta\mathscr{N}^p_{G_b}(\kappa,\mu,\nu), \tag{80}$$

which is a contradiction. Therefore, $\mathscr{P}\mathscr{Q}$ is a p-hybrid contraction mapping on $\mathscr{W}$.

Hence, κ is a coincidence fixed point of $\mathscr{P}$ and $\mathscr{Q}$ and also a solution to integral equation (72) and a solution of the nonlinear fractional differential equation (69). Therefore, we can conclude that all the hypotheses given in Theorem 20 and Theorem 25 are satisfied. Hence, the proof is completed. □

Data Availability

No data were used to support this study.

Additional Points

Code Availability. No code was used in this paper. *Rights and Permissions*. Open access. This article is distributed under the terms of the Creative Commons attribution.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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