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Research Article

Fixed Point Results via Real-Valued Function Satisfying Integral Type Rational Contraction

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In this article, we mainly discuss the existence and uniqueness of fixed point satisfying integral type contractions in complete metric spaces via rational expression using real-valued functions. We improve and unify many widely known results from the literature. Among these, the work of Rakotch (1962), Branciari (2002), and Liu et al. (2013) is extended. Finally, we conclude with an example presented graphically in favour of our work.

1. Introduction

We start this section by recalling the definition of Lebesgue-integrable function. Notify L as a function defined as

$$L = \{l : R^{+} \longrightarrow R^{+}\}, \tag{1}$$

which is nonnegative, summable on each compact subset of R^{+}, and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} l(m)dm > 0. \tag{2}$$

Branciari [1] in 2002 independently and essentially deduced the following result, as an extension of most famous problem of Banach in 1922.

Theorem 1 [1]. *Let (P, d) be a complete metric space, $0 < s < 1$, and $U : P \longrightarrow P$ is a map. If for each $g, h \in P$*

$$\int_0^{d(Ug,Uh)} l(m)dm \le s\int_0^{d(g,h)} l(m)dm, \tag{3}$$

where $l \in L$. Then, $z \in P$ is a unique fixed point of U.

Rhoades [2] made a major extension in 2003 of Branciari [1] by proving a more genral result. His proof introduced a number of interesting ideas for other reseachers to study on integral type of contracions. Prior to Branciari [1] works, Kumar et al. [3] had been able to derive Jungck's [4] fixed point result in sense of integral type contractions. Mocanu and Popa [5] proposed following lemmas that are useful for deriving our main theorem.

Lemma 1 [5]. *Let $l \in L$ and $(r_s)_{s\in N}$ be a nonnegative sequence with $\lim_{s\longrightarrow\infty} r_s = c$ then*

$$\lim_{s\longrightarrow\infty}\int_0^{r_s} l(m)dm = 0 \text{ implies } \lim_{s\longrightarrow\infty} r_s = 0. \tag{4}$$

Lemma 2 [5]. *Let $l \in L$ and $(r_s)_{s\in N}$ be a nonnegative sequence with $\lim_{s\longrightarrow\infty} r_s = c$ then*

$$\lim_{s\longrightarrow\infty}\int_0^{r_s} l(m)dm = \int_0^{c} l(m)dm. \tag{5}$$

Further, Liu et al. [6] extended Branciari's work by including real-valued function and improved the result of Rakotch [7].

Theorem 2 [6]. *Let U be a self-mapping on a complete metric space (P, d) satisfying*

$$\int_0^{d(Ug,Uh)} l(m)dm \leq \delta(d(g,h)) \int_0^{d(g,h)} l(m)dm, \quad (6)$$

for each $g, h \in P$, $\delta : (0,\infty) \longrightarrow [0, 1)$ s.t.

$$\lim \sup \delta(s) < 1, \forall s > 0, \quad (7)$$

and $l \in L$. Then $Pz = z$ for all $z \in U$.

In addition to previous findings, Gupta et al. [8] in 2012 proposed a work for 2 compatible self-maps and derived a result satisfying integral type contraction. In contrast to Rakotch's result, further in 2013, Gupta and Mani [9] placed a rational contraction using real-valued function and established their theorem.

Theorem 3 [9]. *Let U be a self-map on a complete metric space (P, d). If for each $g, h \in P$*

$$\int_0^{d(Ug,Uh)} l(m)dm \leq \gamma(d(g,h)) \int_0^{w(g,h)} l(m)dm, \quad (8)$$

where

$$w(g,h) = \max \left\{ \frac{d(g,Ug) \cdot d(h,Uh)}{d(g,h)}, d(g,h) \right\}, \quad (9)$$

$l \in L$ and a function $\gamma : (0,\infty) \longrightarrow [0, 1)$ with $\lim_{s \longrightarrow n} \sup \gamma(s) < 1$ for all $n > 0$. Then, U has a unique fixed point in P.

About the same time, Liu et al. [10] come with different approach and set up three distinct results for integral type contractions. These studies further give other aspects of integral contractions for researchers, in particular related problems on real-valued functions. Some motivated results on integral type contractions and in metric spaces are refer to see [11–17].

This article is devoted to state the theorem containing real-valued function and to prove the theorem satisfying integral type rational contraction. Our finding extends and generalized some renowned result. An example with graphical representation has been given in favour of our work.

2. Fixed Point via Rational Contraction and by Using Real-Valued Function

Geraghty [18] defined the following class of test function which is more general than the Rakotch [7].

Definition 1 [18]. Define $S = \{\gamma | \gamma : [0,\infty) \longrightarrow [0, 1)\}$ satisfies the condition

$$\gamma(t_i) \longrightarrow 1 \text{ implies } t_i \longrightarrow 0. \quad (10)$$

Example 1. Define the function

$$\gamma(t) = \frac{\sqrt{\log [t^2 + 1]}}{t}, \quad (11)$$

Clearly, $\gamma(t_i) < 1$ and $\gamma(t_i) \longrightarrow 1$ implies $t_i \longrightarrow 0$.

Theorem 4. *Let U be a self-mapping on a complete metric space (P, d) and are such that for each $g, h \in P$*

$$\int_0^{d(Ug,Uh)} l(m)dm \leq \gamma(d(g,h)) \int_0^{w(g,h)} l(m)dm, \quad (12)$$

where

$$w(g,h) = \max \left\{ d(g,h), \frac{d(g,Ug)d(h,Uh)}{1 + d(g,h)}, \frac{d(h,Uh)d(g,Uh)}{1 + d(Ug,Uh)} \right\}, \quad (13)$$

$l \in L$ and $\gamma \in S$. Then U has a unique fixed point.

Proof. Set initial approximation $g_0 \in P$ as an any arbitrary point in P. In general, construct $\{g_i\}$ in P such that

$$Ug_i = g_{i+1}. \quad (14)$$

First, we assert that $\lim_{i \longrightarrow \infty} d(g_i, g_{i+1}) = 0$.
From Equation (12), $\forall i \geq 0$, we have

$$\int_0^{d(g_i,g_{i+1})} l(m)dm \leq \gamma(d(g_{i-1}, g_i)) \int_0^{w(g_{i-1},g_i)} l(m)dm, \quad (15)$$

where

$$\begin{aligned} w(g_{i-1}, g_i) &= \max \left\{ d(g_{i-1}, g_i), \frac{d(g_{i-1}, Ug_{i-1})d(g_i, Ug_i)}{1 + d(g_{i-1}, g_i)}, \frac{d(g_i, Ug_{i-1})d(g_{i-1}, Ug_i)}{1 + d(Ug_{i-1}, Ug_i)} \right\}, \\ &= \max \left\{ d(g_{i-1}, g_i), \frac{d(g_{i-1}, g_i)d(g_i, g_{i+1})}{1 + d(g_{i-1}, g_i)}, \frac{d(g_i, g_i)d(g_{i-1}, g_{i+1})}{1 + d(g_i, g_{i+1})} \right\}, \\ &= \max \{ d(g_{i-1}, g_i), d(g_i, g_{i+1}) \}. \end{aligned} \quad (16)$$

Now, if $d(g_i, g_{i+1}) > d(g_{i-1}, g_i)$, then $w(g_{i-1}, g_i) = d(g_i, g_{i+1})$. Hence, from Equation (15) and using the fact that $\gamma \in S$, we arrived at a contradiction. Therefore, $d(g_i, g_{i+1}) < d(g_{i-1}, g_i)$ and so $w(g_{i-1}, g_i) = d(g_{i-1}, g_i)$.

Thus, Equation (15) implies that

$$\int_0^{d(g_i,g_{i+1})} l(m)dm \le \gamma(d(g_{i-1}, g_i)) \int_0^{d(g_{i-1},g_i)} l(m)dm. \tag{17}$$

Since $\gamma \in S$, we have

$$\int_0^{d(g_i,g_{i+1})} l(m)dm \le \int_0^{d(g_{i-1},g_i)} l(m)dm. \tag{18}$$

Similarly,

$$\int_0^{d(g_{i-1},g_i)} l(m)dm \le \int_0^{d(g_{i-2},g_{i-1})} l(m)dm. \tag{19}$$

Thus, a monotone decreasing sequence $\{\int_0^{d(g_i,g_{i+1})} l(m) dm\}$ of nonnegative reals has obtained, and so there exists $s \ge 0$ such that

$$\lim_{i\longrightarrow\infty} \int_0^{d(g_i,g_{i+1})} l(m)dm = s. \tag{20}$$

Assume that $s > 0$. Letting $i \longrightarrow \infty$ in Equation (15) and using Equation (20), we get $s \le s$, as $\gamma \in S$, a contradiction, implies $s = 0$ and hence

$$\lim_{i\longrightarrow\infty} \int_0^{d(g_i,g_{i+1})} l(m)dm = 0. \tag{21}$$

Lemma 1 implies

$$\lim_{i\longrightarrow\infty} d(g_i, g_{i+1}) = 0. \tag{22}$$

Next, we assert that sequence $\{g_i\}$ is Cauchy.

Assume that for an $\epsilon > 0$, there exists subsequences $\{g_{i_s}\}$ and $\{g_{w_s}\}$ of $\{g_i\}$ with $w_s > i_s \ge s, s > 0$ satisfying

$$d\left(g_{w_s}, g_{i_s}\right)\Big) \ge \epsilon \text{ and } d\left(g_{w_{s-1}}, g_{i_s}\right) < \epsilon, \tag{23}$$

$\forall s \ge 0$, consider

$$\begin{aligned}\int_0^{\varepsilon} l(m)dm &\le \int_0^{d\left(g_{w_s},g_{i_s}\right)} l(m)dm \\ &\le \gamma\left(\left(d\left(g_{w_{s-1}}, g_{i_{s-1}}\right)\right)\right) \int_0^{w\left(g_{w_{s-1}},g_{i_{s-1}}\right)} l(m)dm,\end{aligned} \tag{24}$$

where

$$w\left(g_{w_{s-1}}, g_{i_{s-1}}\right) = \max\left\{d\left(g_{w_{s-1}}, g_{i_{s-1}}\right), \frac{d\left(g_{w_{s-1}}, g_{w_s}\right) d\left(g_{i_{s-1}}, g_{i_s}\right)}{1 + d\left(g_{w_{s-1}}, g_{i_{s-1}}\right)}, \frac{d\left(g_{i_{s-1}}, g_{w_s}\right) d\left(g_{w_{s-1}}, g_{i_s}\right)}{1 + d\left(g_{w_s}, g_{i_{s-1}}\right)}\right\}. \tag{25}$$

Triangle inequality implies that

$$d\left(g_{w_{s-1}}, g_{i_{s-1}}\right) = d\left(g_{w_{s-1}}, g_{i_s}\right) + d\left(g_{i_s}, g_{i_{s-1}}\right). \tag{26}$$

From Equation (23), $d(g_{w_{s-1}}, g_{i_{s-1}}) \le \epsilon + d(g_{i_s}, g_{i_{s-1}})$. On taking limits and using Equations (22), we get

$$\lim_{i\longrightarrow\infty} d\left(g_{w_{s-1}}, g_{i_{s-1}}\right) = \epsilon. \tag{27}$$

Again, we know

$$d\left(g_{i_{s-1}}, g_{w_s}\right) \le d\left(g_{i_{s-1}}, g_{w_{s-1}}\right) + d\left(g_{w_{s-1}}, g_{w_s}\right). \tag{28}$$

Therefore, on taking $\lim_{s\longrightarrow\infty}$ and using (23) and (22), we get

$$\lim_{s\longrightarrow\infty} d\left(g_{i_{s-1}}, g_{w_s}\right) = \epsilon. \tag{29}$$

Hence, from Equation (25), on taking $\lim_{s\longrightarrow\infty}$ and using Equations (23), (27), and (29), we get

$$\lim_{s\longrightarrow\infty} w\left(g_{w_{s-1}}, g_{i_{s-1}}\right) = \epsilon. \tag{30}$$

Thus, on letting $\lim_{s\longrightarrow\infty}$, Equation (24) implies that

$$\int_0^{\varepsilon} l(m)dm \le \gamma(\epsilon) \int_0^{\epsilon} l(m)dm, \tag{31}$$

where we arrived at a contradiction as $\gamma \in S$. Therefore, sequence $\{g_i\}$ is Cauchy. Call a limit ν such that from (12)

$$\lim_{i\longrightarrow\infty} g_{i+1} = \lim_{i\longrightarrow\infty} U g_i = \nu. \tag{32}$$

Now, assert that ν is a fixed point of U.

Indeed, continuity of U implies that

$$\nu = \lim_{i\longrightarrow\infty} U g_i = U \lim_{i\longrightarrow\infty} g_i = U\nu. \tag{33}$$

Secondly, assume U is not continuous and also let $Uv \neq v$. Then, clearly $d(Uv, v) \neq 0$. Assume that $d(Uv, v) > 0$. Therefore

$$0 < \int_0^{d(Uv,v)} l(m)dm = \lim_{i \longrightarrow \infty} \int_0^{d(Uv,g_{i+1})} l(m)dm \leq \lim_{i \longrightarrow \infty} \int_0^{w(v,g_i)} l(m)dm, \tag{34}$$

where

$$w(v, g_i) = \max\left\{ d(v, g_i), \frac{d(v, Uv)d(g_i, Ug_i)}{1 + d(v, g_i)}, \frac{d(g_i, Uv)d(v, Ug_i)}{1 + d(Uv, Ug_i)} \right\}. \tag{35}$$

Take $\lim_{i\longrightarrow\infty}$, we obtain

$$\lim_{i \longrightarrow \infty} w(v, g_i) = \max\left\{ d(v, v), \frac{d(v, Uv)d(v, v)}{1 + d(v, Ug_i)}, \frac{d(v, Uv)d(v, v)}{1 + d(Uv, v)} \right\}, = 0. \tag{36}$$

Hence from (34),

$$0 < \int_0^{d(Uv,v)} l(m)dm = \int_0^{d(Uv,Ug_i)} l(m)dm \leq 0, \tag{37}$$

Therefore, $Uv = v$.

For uniqueness, assume there exist a point $s \neq v$ other than v s.t $d(Us, s) = 0$. Consider,

$$\int_0^{d(s,v)} l(m)dm = \int_0^{d(Us,Uv)} l(m)dm \leq \gamma(d(s, v)) \int_0^{w(s,v)} l(m)dm, \tag{38}$$

where

$$\begin{aligned} w(s, v) &= \max\left\{ d(s, v), \frac{d(s, Uv)d(v, Uv)}{1 + d(s, v)}, \frac{d(v, Us)d(s, Uv)}{1 + d(Us, Uv)} \right\}, \\ &= \max\left\{ d(s, v), \frac{d(s, s)d(v, v)}{1 + d(s, v)}, \frac{d(v, s)d(s, v)}{1 + d(s, v)} \right\}. \end{aligned} \tag{39}$$

Since $d(s, v)/1 + d(s, v) < 1$, therefore

$$w(s, v) \leq \max\{d(s, v), 0, d(s, v)\} \leq d(s, v). \tag{40}$$

Using the fact that $\gamma \in S$ and from (38), we have

$$\int_0^{d(s,v)} l(m)dm < \int_0^{d(s,v)} l(m)dm. \tag{41}$$

This implies $s = v$, and hence, fixed point of U is unique. This accomplished our proof. □

Theorem 5. *Let a self-map U on a complete metric space (P, d) such that for each $g, h \in P$*

$$\int_0^{d(Ug,Uh)} l(m)dm \leq \delta(d(g, h)) \int_0^{d(g,h)} l(m)dm + \gamma(d(g, h)) \int_0^{i(g,h)} l(m)dm, \tag{42}$$

where

$$i(g, h) = \max\left\{ \frac{d(g, Ug)d(h, Uh)}{1 + d(g, h)}, \frac{d(h, Uh)d(g, Uh)}{1 + d(Ug, Uh)} \right\}. \tag{43}$$

$l \in L$ and $\delta, \gamma \in S$ with $\delta(m) + \gamma(m) < 1$. Then, U has a unique fixed point.

Proof. Since $\delta, \gamma \in S$ with $\delta(m) + \gamma(m) < 1$. Let $\gamma(m) = \max\{\delta(m), \gamma(m)\}$. Then from Equation (42), we have

$$\begin{aligned} \int_0^{d(Ug,Uh)} l(m)dm &\leq \delta(d(g, h)) \int_0^{d(g,h)} l(m)dm + \gamma(d(g, h)) \int_0^{i(g,h)} l(m)dm \\ &< \gamma(d(g, h)) \int_0^{\max\{d(g,h), i(g,h)\}} l(m)dm \\ &\leq \gamma(d(g, h)) \int_0^{w(g,h)} l(m)dm, \end{aligned} \tag{44}$$

where

$$w(g, h) = \max\left\{ d(g, h), \frac{d(g, Ug)d(h, Uh)}{1 + d(g, h)}, \frac{d(h, Uh)d(g, Uh)}{1 + d(Ug, Uh)} \right\}. \tag{45}$$

Rest of the proof is on the same line of Theorem 4. □

If we take $l(t) = 1$, then we have the following two consequence results from our main theorem

Corollary 1. *Let a self-map U on a complete metric space (P, d) such that for each $g, h \in P$*

$$d(Ug, Uh) \leq \gamma(d(g, h))w(g, h), \tag{46}$$

where

$$w(g, h) = \max\left\{ d(g, h), \frac{d(g, Ug)d(h, Uh)}{1 + d(g, h)}, \frac{d(h, Uh)d(g, Uh)}{1 + d(Ug, Uh)} \right\}, \tag{47}$$

and $\gamma \in S$. Then, U has a unique fixed point.

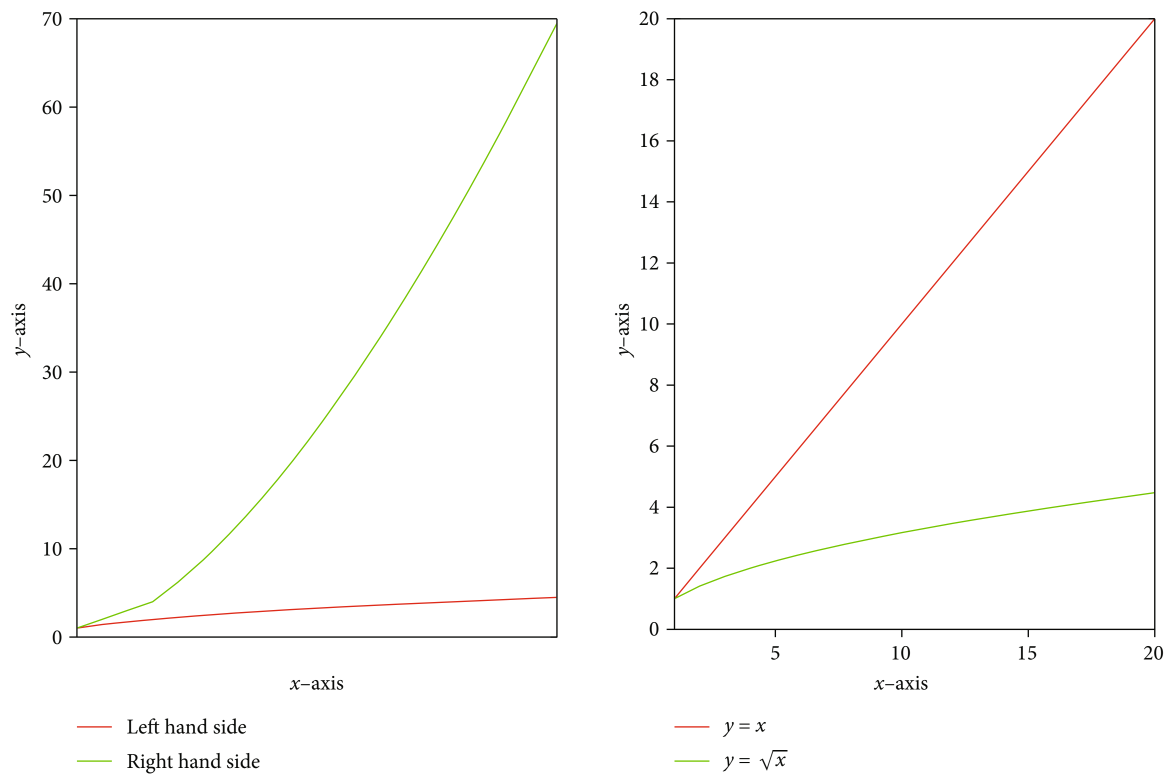

FIGURE 1: Graphical representation of inequality satisfying Example 2. Also showing the graph of curves $y = \sqrt{x}$ and $y = x$ with fixed point $x = 1$.

Corollary 2. *Let a self-map U on a complete metric space (P, d) such that for each $g, h \in P$*

$$d(Ug, Uh) \leq \delta(d(g, h))d(g, h) + \gamma(d(g, h))i(g, h), \quad (48)$$

where

$$i(g, h) = \max\left\{\frac{d(g, Ug)d(h, Uh)}{1 + d(g, h)}, \frac{d(h, Uh)d(g, Uh)}{1 + d(Ug, Uh)}\right\}, \quad (49)$$

and $\delta, \gamma \in S$ with $\delta(t) + \gamma(t) < 1$. Then U has a unique fixed point.

Example 2. Let $P = N$. Define the metric $d(g, h) = \max\{g, h\}$, for all $g, h \in P$. Clearly, (P, d) is a metric space. Define a function $U : P \longrightarrow P$ as $U(g) = \sqrt{g}, \forall g \in P$. Also defined $l \in L$ as $l(m) = 2m, \forall m \in R^+$ and $\delta : R^+ \longrightarrow [0, 1)$ is defined by $\delta(m) = 9/10$

Figure 1 showing the plot of inequality (12) satisfying the Example 2. Thus, all the conditions of Theorem 4 are satisfied. Clearly, $1 \in P$ is a fixed-point of U.

Remark 1. Theorem 4 and Theorem 5 are unified and extended results of Liu et al. [10] and Branciari [1].

Remark 2. Corollary 2 is an extension of the result of Rakotch [7] with more general test functions.

Remark 3. In Theorem 5, on letting $\gamma(t) = 0$, we obtain the result of 1.2 (result of Liu et al. [10]).

Remark 4. If we take $\gamma(t) = 0$ in Corollary 2, we deduc the result of Geraghty [18].

3. Conclusion

We conclude this note by mentioning that our proved result is a further extension of Branciari result into other settings. Some remarks and an example are given to justify that our results are extension and generalized version of some known results of literature.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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