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Research Article

A Modified RBF Collocation Method for Solving the Convection-Diffusion Problems

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The main purposes of this study are to propose the modified radial basis function (RBF) collocation method using a hybrid radial basis function to solve the convection-diffusion problems numerically and to choose the optimal shape parameter of radial basis functions. The modified numerical scheme is tested on a benchmark problem with varying shape parameters. The root mean square error and maximum error are used to validate the accuracy and efficiency of the method. The proposed method can be a good alternative to the radial basis function collocation method to improve accuracy and results.

1. Introduction

The convection-diffusion problem is important in many branches of science and engineering governed by the convection-diffusion equation [1–5]. The convection-diffusion equation is a fundamental equation that combines convection and diffusion processes to represent the problem process. When dealing with complex geometry, the analytical method is difficult to use, but the numerical methods have been tackled [6–9]. However, it is well known that the solution to this problem becomes oscillatory when the convection-diffusion problem becomes a convection-dominated problem, meaning the coefficient in the diffusion part is very small. In this convection-dominated problem, a thin boundary layer (there is a very high gradient) is usually formed if the standard numerical procedure is performed without special treatments. Standard numerical methods are also limited in the geometric domain. Moreover, the solution to this problem presents a challenging computational task.

The numerical methods known as radial basis function (RBF) collocation methods have been proposed and attracted by researcher because these methods do not require a mesh generation to discretize the problem domains [10–19]. Typically, these methods are based on the collocation techniques, and the computing process of the methods is efficient and accurate when only the Dirichlet boundary conditions exist [20]. However, for a large number of collocation points in a domain, these methods may have a major shortcoming when they are applied to solve the Neumann boundary conditions. Furthermore, the computing process leads to the ill-conditioned problem, and several approaches have been proposed to solve this problem [21–33]. A main tool of the RBF collocation techniques is RBF, which is used to approximate the solution of the partial differential equations (PDEs). Chuathong et al. [34, 35] have proposed the effectiveness of several well-known and mostly used RBFs to solve the nonlinear class of partial differential equations. In addition, it is found that the Cubic Matérn RBF type produces "the best results" quality for all test cases, whereas the Gaussian RBF produces "worst results" quality. For the interpolation of large points, however, the typical RBF forms lead to large and ill-conditioned systems. To avoid this limitation, Mishra et al. [36] have proposed a new RBF that combines the Gaussian and cubic types. They found that the proposed RBF significantly improved the condition number of the matrix system. Also, the ill-conditioned problems are treated by Kansa's method using the proposed RBF for application in several benchmark problems: 1D, 2D, and 3D. Recently, this method using a hybrid RBF has gained popularity in many fields of science and engineering, and it has been used to solve large data approximations [37–39]. This immediately raises the question of whether this newly invented RBF can be applied to the problem of solving PDEs, specifically the convection-diffusion equation. This motivates the author to modify the RBF collocation technique in order to solve the convection-diffusion problem and alleviate the ill-conditioned problems. Furthermore, the convection-dominated problem is expected to be solved more effectively.

In this article, the newly invented RBF [36] is applied to solve the convection-diffusion problem. While aspects of the hybrid RBF have been discovered, it is still mentioned in selecting the optimal shape parameter in the Gaussian part, which is not straightforward. This means that the choice of the shape parameter is specific. To remedy this case, the new hybrid is modified by adding weight to the Gaussian and cubic RBFs. The two RBFs are then combined. The weight is added to control the contribution of the Gaussian and cubic parts. Several possible combinations of both RBFs are investigated. The optimal weight is chosen in the interval $(0, 1)$ using the minimum root mean square error. Solving the convection-diffusion problem using some traditional RBF collocation methods without special treatments is quite ill-conditioned; therefore, a new modified hybrid RBF on the RBF collocation scheme is proposed to solve the problems. As shown in the numerical results, applying the modified RBF collocation method significantly reduces the error in solving the convection-diffusion problem, especially when the diffusion coefficient is small.

The article is organized as follows. In Section 2, a brief conventional RBF collocation method for solving partial differential equations is introduced. In Section 3, the invented hybrid RBF of Mishra et al. [36] is introduced, and the difference between the Gaussian and cubic RBFs is briefly discussed. The advantages of both RBFs are employed to modify the hybrid RBF, and the traditional RBF collocation method is also modified for the convection-diffusion problem using the new modified hybrid RBF. In Section 4, to validate the proposed approach, the proposed RBF collocation method is applied to solve the convection-diffusion problem. Moreover, the results obtained by this method are compared to those obtained by exact solutions. Furthermore, to alleviate the convection-dominated problems and the difficulty of conditioning, the modified RBF collocation method using the various weights of the hybrid RBF is studied for different diffusion coefficients. Finding the optimal shape parameter of the Gaussian in a hybrid RBF is also discussed to obtain the best results. A short conclusion is drawn in Section 5.

2. The Radial Basis Function Collocation Methods (RBF Collocation Methods)

Before introducing the proposed modified RBF collocation scheme, a brief review of the RBF collocation methods is given. To illustrate the basic idea, the following two-dimensional elliptic partial differential equations are considered:

$$Lu(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \text{ in } \Omega, \tag{1}$$

$$Bu(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \text{ on } \partial\Omega, \tag{2}$$

where L and B represent the elliptical differential operators in a bounded domain Ω and a boundary domain $\partial\Omega$,, respectively.

The basic idea of the RBF collocation methods is to apply the RBF ϕ to assume the approximate solution $\hat{u}$ of (1) and (2) provided by

$$\hat{u}(\mathbf{x}) = \sum_{j=1}^{N} c_j \phi(\|\mathbf{x} - \mathbf{x}_j\|), \tag{3}$$

where $\|\cdot\|$ is the Euclidean norm and c_j are the unknown coefficients.

Let $\{\mathbf{x}_1, \mathbf{x}_2 \cdots, \mathbf{x}_N\}$ denote the set of collocation points, $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{N_I}\}$ denote the set of interior points, and $\{\mathbf{x}_{N_I+1}, \mathbf{x}_{N_I+2}, \cdots, \mathbf{x}_N\}$ denote the set of boundary points where N_I is the number of interior points and N is the total number of collocation points.

Using (3), the governing equations (1) and (2) become

$$L\hat{u}(\mathbf{x}) = \sum_{j=1}^{N_I} c_j L\, \phi(\|\mathbf{x} - \mathbf{x}_j\|) = f(\mathbf{x}), \tag{4}$$

$$B\hat{u}(\mathbf{x}) = \sum_{j=N_I+1}^{N} c_j B\phi(\|\mathbf{x} - \mathbf{x}_j\|) = g(\mathbf{x}), \tag{5}$$

where N_B is the number of boundary points and $N = N_I + N_B$. Therefore, applying the collocation points to (4) and (5), we can obtain $\{c_j\}, j = 1, 2, 3, \cdots, N$ through (4) and (5). Then, we can obtain the approximate solution to (3) at all given points.

3. The Modified RBF Collocation Method

In this section, we apply a hybrid RBF [36] concept to the RBF collocation methods. RBFs are the main tools used in RBF collocation methods to approximate the unknown function using known data points. The RBF is defined as follows.

Definition 1. A function $\varphi : \mathbb{R}^N \longrightarrow \mathbb{R}$ is said to be radial if there exists a univariate function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that $\varphi(\mathbf{x}) = \phi(r)$, $r = \|\mathbf{x}\|$, where $\|\cdot\|$ represents the Euclidean norm. Typical RBF and their expression are shown in Table 1.

The stability and accuracy of the RBF interpolation depend on the aspects of the algorithm and the data involved.

Table 1: Typical RBF and their expression.

RBFs	Mathematical expression
Multiquadric	$\phi(r) = \left(1 + (\varepsilon r)^2\right)^{1/2}$
Inverse multiquadric	$\phi(r) = \left(1 + (\varepsilon r)^2\right)^{-1/2}$
Gaussian	$\phi(r) = e^{-(\varepsilon r)^2}$
Thin plate spline	$\phi(r) = r^2 \log(r)$
Cubic	$\phi(r) = r^3$
Wendland's	$\phi(r) = \left(1 + (\varepsilon r)^2\right)_+^4 (4\varepsilon r + 1)$

Mishra et al. [36] have proposed a hybrid radial basis function that combines the conventional Gaussian and a cubic RBF, as given by

$$\phi\left(r_j\right) = e^{-\left(\varepsilon r_j\right)^2} + r_j^{\,3}, \tag{6}$$

where $r_j = \|\mathbf{x} - \mathbf{x}_j\|$, $\mathbf{x} \in \mathbb{R}^2$ are in a domain, and $\mathbf{x}_j$ are the collocation center points for $j = 1, 2, \cdots, N$.

The first part of equation (6) is the Gaussian RBF which is dependent on the shape parameter ε, and the second one is the cubic RBF which does not contain the shape parameter. This combination of RBFs makes "the Gaussian RBF with small cubic doping" for large shape parameter. On the other hand, for the small shape parameter, the cubic term dominates the RBF, making "a cubic RBF with small Gaussian doping." However, it is noted that the first part of equation (6) still depends on the shape parameter which is not straightforward in practical use meaning that it is often selected in an "ad hoc" manner. For this reason, this investigation is aimed at proposing a newly modified version of the hybrid RBF. Formula (6) is now further modified to improve the results. It is designed to rely on an additional weighting function, α. The following is the modified form:

$$\phi\left(r_j\right) = (1 - \alpha) e^{-\left(\varepsilon r_j\right)^2} + \alpha r_j^{\,3}, \tag{7}$$

where $0 < \alpha < 1$.

The weight α is added to the hybrid RBF to control the contribution of the Gaussian and cubic parts. In addition, the accuracy of the hybrid RBF is affected by the type of problem and the collocation points. The weight α helps an ill-conditioned problem when the hybrid RBF uses a small shape parameter and alleviates the singular matrix in cubic part. This means that the collocation method using the hybrid RBF will produce better results in terms of accuracy. The optimal shape parameter ε and the weight α are chosen according to the minimum root mean square error.

Moreover, this work is expanded to another challenging problem of solving the convection-diffusion problem which has piqued the interest of many researchers. In addition, the proposed hybrid RBF is used to modify the RBF collocation methods. First, consider the convection-diffusion problem [40] as follows:

$$L(u) = \mathbf{v}^T \cdot \nabla u - \nabla^T (\mathbf{D} \nabla u) + \beta u - q(\mathbf{x}) = 0 \text{ in } \Omega, \tag{8}$$

with the Neumann boundary condition on boundary

$$B(u) = \mathbf{n}^T \mathbf{D} \nabla u + \bar{q}_n = 0 \text{ on } \partial\Omega_1, \tag{9}$$

and the Dirichlet boundary condition on boundary

$$u - \bar{u} = 0 \text{ on } \partial\Omega_2. \tag{10}$$

Let $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\}$ denote a set of collocation points in a domain $\Omega \cup \partial\Omega_1 \cup \partial\Omega_2$, $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{N_I}\}$ denote a set of interior points, and $\{\mathbf{x}_{N_I+1}, \mathbf{x}_{N_I+2}, \cdots, \mathbf{x}_N\}$ denote a set of boundary points on $\partial\Omega_1 \cup \partial\Omega_2$.

Assume the following approximate solution to the convection-diffusion problems (8) and (9):

$$\widehat{u}(\mathbf{x}) = \sum_{j=1}^{N} c_j \phi\left(\left\|\mathbf{x} - \mathbf{x}_j\right\|\right), \tag{11}$$

where ϕ is a hybrid RBF and N is the total number of collocation points in a domain $\Omega \cup \partial\Omega_1 \cup \partial\Omega_2$.

The coefficients $\{c_1, c_2, \cdots, c_N\}$ in equation (11) are determined by using the collocation points through (8), (9), and (10) as follows:

$$L(\widehat{u}_i) = \mathbf{v}^T \cdot \nabla \widehat{u}_i^+ - \nabla^T (\mathbf{D} \nabla \widehat{u}_i) + \beta \widehat{u}_i - q(\mathbf{x}_i) = 0, \tag{12}$$

$$B_1(\widehat{u}_i) = \mathbf{n}^T \mathbf{D} \nabla \widehat{u}_i + \bar{q}_n = 0, \tag{13}$$

$$B_2(\widehat{u}_i) = \widehat{u}_i - \bar{u} = 0. \tag{14}$$

From (12), (13), and (14), we have the following matrix system:

$$\boldsymbol{\psi} \mathbf{c} = \mathbf{f}, \tag{15}$$

where $\boldsymbol{\psi}_{ij} = \phi(\|\mathbf{x}_i - \mathbf{x}_j\|)$, $i, j = 1, 2, \cdots, N$ and $\mathbf{c} = [c_1, c_2, \cdots, c_N]^T$.

The unknown coefficient matrix $\mathbf{c}$ is obtained by solving the system (15). Hence, for any $\mathbf{x}_i \in \Omega$, the approximate solution $\widehat{u}(\mathbf{x}_i)$ can be found in equation (11).

Next, it is observed that

$$L\widehat{u}_i = L\widehat{u}(\mathbf{x}_i) = \sum_{j=1}^{N} c_j L\phi\left(\left\|\mathbf{x}_i - \mathbf{x}_j\right\|\right) = L\boldsymbol{\psi}\mathbf{c}, \quad i = 1, 2, \cdots, N_I, \tag{16}$$

$$B\widehat{u}_i = B\widehat{u}(\mathbf{x}_i) = \sum_{j=1}^{N} c_j B\phi\left(\left\|\mathbf{x}_i - \mathbf{x}_j\right\|\right) = B\boldsymbol{\psi}\mathbf{c}, \quad i = N_I + 1, \cdots, N, \tag{17}$$

where $L\boldsymbol{\psi}$ and $B\boldsymbol{\psi}$ defined in (16) and (17) are the same as equations (4) and (5), respectively.

4. Numerical Results

To illustrate the effectiveness of the proposed approach, the convection-diffusion equation is applied. Throughout this section, HybRBF denotes the hybrid RBF in (6), MHybRBF

Table 2: Comparison of results obtained by applying the RBF collocation method with GA, CB, HybRBF, and MHybRBF with different ρ.

ρ	GA RMSE	GA MAXE	GA CN	Time	CB RMSE	CB MAXE	CB CN	Time
100	3.615045e-06	6.990812e-06	1.537902e+23	0.123355	1.470723e-07	3.059453e-07	4.684657e+11	0.116703
10	3.476547e-04	6.694427e-04	1.304255e+22	0.116527	1.428050e-05	3.034218e-05	5.151761e+10	0.179804
1	2.280170e-02	4.270190e-02	1.026336e+21	0.132901	1.109043e-03	2.366442e-03	1.191804e+10	0.122082
0.5	5.552731e-02	1.055118e-01	1.724032e+21	0.132075	3.293163e-03	7.376807e-03	1.054947e+10	0.122629
0.1	3.533230e-01	8.452437e-01	5.881338e+20	0.123835	5.047532e-02	1.422299e-01	6.712422e+10	0.118474
0.01	5.981820e-01	2.266399e+00	1.198877e+21	0.125228	3.421569e-01	1.458199e+00	3.078447e+11	0.125130
0.001	2.114194e-01	1.248382e+00	3.568844e+20	0.126839	1.732781e+01	7.510594e+01	1.602582e+13	0.124880
0.0001	4.240498e-01	2.052057e+00	6.242204e+21	0.114314	1.062712e+00	8.896234e+00	3.631293e+11	0.146026

ρ	HybRBF RMSE	HybRBF MAXE	HybRBF CN	Time	MHybRBF RMSE	MHybRBF MAXE	MHybRBF CN	Time
100	7.705245e-06	1.434408e-05	7.463696e+12	0.122474	8.164123e-07	1.778744e-06	7.935031e+11	0.121252
10	6.449282e-04	1.200222e-03	7.820077e+11	0.134633	8.006966e-05	1.725578e-04	8.404555e+10	0.128443
1	2.337358e-03	4.480407e-03	1.454403e+11	0.140067	6.479810e-03	1.319792e-02	1.359125e+10	0.164191
0.5	5.209617e-01	1.127411e+00	1.149173e+11	0.132664	2.108005e-02	4.226807e-02	9.773317e+09	0.116316
0.1	5.209617e-01	1.127411e+00	3.183265e+11	0.114587	1.658683e-01	3.137132e-01	1.585630e+10	0.120703
0.01	3.210199e-01	8.306938e-01	4.371672e+11	0.115497	4.172998e-01	1.540322e+00	2.048141e+12	0.135795
0.001	4.251006e-01	1.577832e+00	4.997906e+11	0.114875	1.235107e+00	4.180278e+00	3.094705e+11	0.117116
0.0001	3.209438e+00	9.752944e+00	8.167756e+11	0.137506	5.994588e-01	2.263910e+00	1.955926e+11	0.118089

denotes the modified RBF in (7), GA denotes the Gaussian RBF, and CB denotes the cubic RBF.

The results obtained by the numerical scheme are compared with those obtained by the exact solution to assess the effectiveness of the proposed approach. The numerical accuracy is measured using the following errors:

$$\text{RMSE} = \frac{1}{\sqrt{N}} \|u(\mathbf{x}) - \hat{u}(\mathbf{x})\|, \qquad \text{MAXE} = \|u(\mathbf{x}) - \hat{u}(\mathbf{x})\|_{\infty}, \tag{18}$$

where RMSE is the root mean square error, MAXE is the maximum error, and $\hat{u}(\mathbf{x})$ is the approximate solution obtained by the proposed method. MATLAB is used to perform on a notebook computer with Intel Core i7 Home Basic 64 bit. Regarding the performance of the proposed method, the CPU times are reported and computed in seconds. Moreover, the stability of the method is evaluated by calculating the condition number (CN) of the collocation matrix $\boldsymbol{\psi}$, using the MATLAB command cond.

In the following implementation, the convection-diffusion problems (9), (10), and (11) are investigated in a rectangular domain $[0,1]\times[0,1]$, the coefficients

$$D = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}, \quad v = \{3-x, 4-y\}, \quad \beta = 1, \tag{19}$$

and ρ is a given constant of diffusion coefficient.

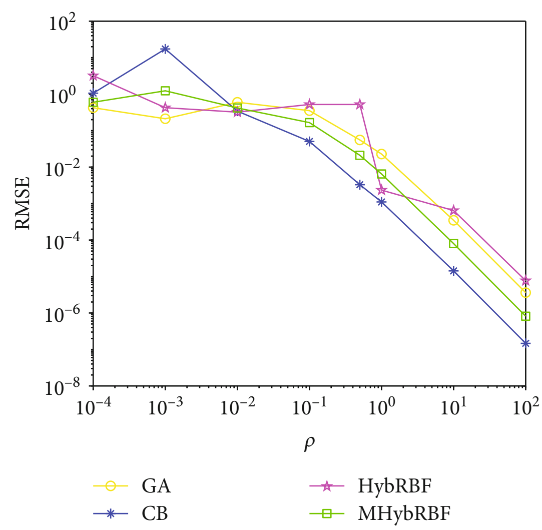


Figure 1: RMSE of the collocation method using GA, CB, HybRBF, and MHybRBF with different ρ.

The boundary conditions are considered as follows:

$$u(0,x) = 0, \quad u(1,y) = 0, \quad u(x,0) = 0, \quad u(x,1) = 1. \tag{20}$$

Table 3: The optimized shape parameter (ε) of HybRBF2 and MHybRBF2 with different ρ.

ρ	HybRBF2					MHybRBF2				
	ε	RMSE	MAXE	CN	Time	ε	RMSE	MAXE	CN	Time
100	2.45	1.538491*e*-08	5.205199*e*-08	1.311588*e*+12	0.169221	3.54	2.910194*e*-07	6.910851*e*-07	8.426454*e*+11	0.121009
10	2.49	1.168458*e*-06	4.199704*e*-06	1.536089*e*+11	0.120400	3.56	2.736761*e*-05	6.770259*e*-05	8.961082*e*+10	0.119983
1	3.00	2.337358*e*-03	4.480407*e*-03	1.454403*e*+11	0.112102	3.93	1.588563*e*-03	3.551225*e*-03	1.539699*e*+10	0.115826
0.5	18.00	3.857760*e*-03	7.847551*e*-03	2.209149*e*+06	0.226041	4.36	3.900101*e*-03	8.252530*e*-03	1.330964*e*+10	0.129819
0.1	9.45	1.294861*e*-03	3.710988*e*-03	9.744350*e*+09	0.113749	10.00	9.266198*e*-03	2.282203*e*-02	6.723776*e*+09	0.117170
0.01	2.71	3.140152*e*-01	7.836397*e*-01	3.216466*e*+11	0.188296	9.38	1.086233*e*-01	4.221994*e*-01	1.067272*e*+10	0.134580
0.001	3.97	1.962120*e*-01	9.497762*e*-01	1.337207*e*+11	0.177430	46.5	7.834417*e*-02	3.383703*e*-01	1.872398*e*+08	0.123130
0.0001	61.10	2.196106*e*-01	9.531780*e*-01	6.056237*e*+10	0.108612	47.59	1.760771*e*-01	1.287923*e*+00	1.014938*e*+08	0.115460

The exact solution for this problem is given by

$$u(x,y)_{\text{exact}} = \sin(x)\left(1 - e^{-(2(1-x))/\rho}\right)\left(1 - e^{-(3(1-y))/\rho}\right). \quad (21)$$

In all case studies, the nodal distributed model is used: 21×21 (441 points) uniformly distributed points in a domain and on the boundary points. In the first case study, the collocation method using GA, CU, HybRBF, and MHybRBF is examined to solve the convection-diffusion problem at different diffusion coefficients ρ. Here, the value of the shape parameter contained in HybRBF, MHybRBF, and GA is often selected as 3, and the weight α of MHybRBF is given as 0.5. Table 2 shows a summary of all results obtained by the collocation method using all four RBFs. It can be observed that the errors obtained by applying each method differ slightly. When the diffusion coefficient ρ is very small, or the problem becomes a convection-dominated problem, the errors of the collocation method using all four RBFs become large and lead to oscillatory results. In terms of stability, the CN of collocation matrix is analyzed, and it is discovered that the CN obtained by using CB, HybRBF, and MHybRBF is similar. The CN has an order of magnitude between 10^{+9} and 10^{+12} as shown in Table 2. Figure 1 also depicts a comparison of the RMSE of the collocation method using all four RBFs with various diffusion coefficients ρ. It can be noted that when the diffusion coefficient ρ is large (i.e., $\rho \geq 1$), the computational effectiveness of these methods can produce good results. However, GA can still perform slightly better than the others when the diffusion coefficient is small (e.g., $\rho = 0.001$). This means that the RBF collocation method using GA can alleviate the convection-dominated problem for choosing the shape parameter as 3. Although GA yields the numerical results that agree with the exact solution for a very small diffusion coefficient, CN obtained by GA is higher than that obtained by CU, HybRBF, and MHybRBF. Therefore, GA is combined with CU to obtain a lower CN which becomes HybRBF. In particular, the comparison of HybRBF and MHybRBF is considered, and it is found that MHybRBF provides slightly better results than HybRBF. As the results shown in Figure 1 and Table 1, the optimal shape parameter is discussed and chosen by using the minimum RMSE to improve the computational effectiveness of the collocation method using HybRBF and MHybRBF with a very small diffusion coefficient, and it can be written in mathematical form as

$$\begin{gathered} \text{Minimize}_{\alpha,\varepsilon}\xi(\alpha,\varepsilon), \\ \text{subject to } 0 < \varepsilon < 100, 0 < \alpha < 1, \end{gathered} \quad (22)$$

where ξ is the objective function computed through the RMSE and α, ε are the parameters in the hybrid RBF.

Let HybRBF2 and MHybRBF2 denote the HybRBF and MHybRBF using the optimal shape parameter, respectively. The value of the shape parameter ε of both RBFs is tested by varying the value between 0 and 100. In this investigation, MHybRBF and MHybRBF2 are computed with a specific $\alpha = 0.5$.

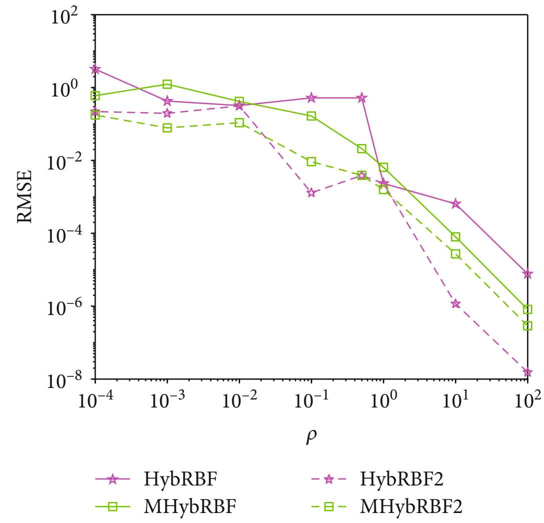


Figure 2: The comparison of RMSE of HybRBF, MHybRBF, HybRBF2, and MHybRBF2 with different ρ.

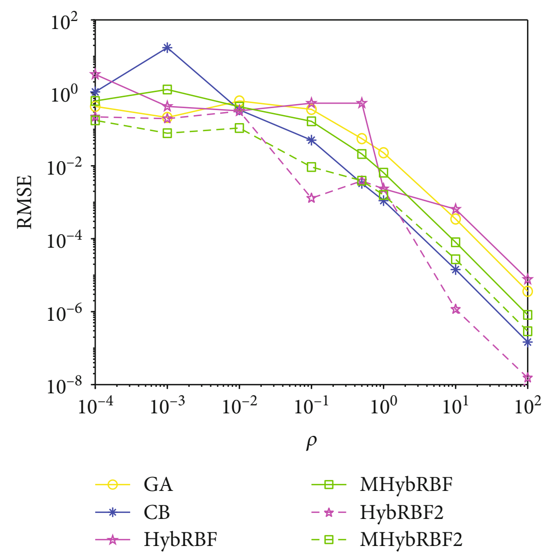
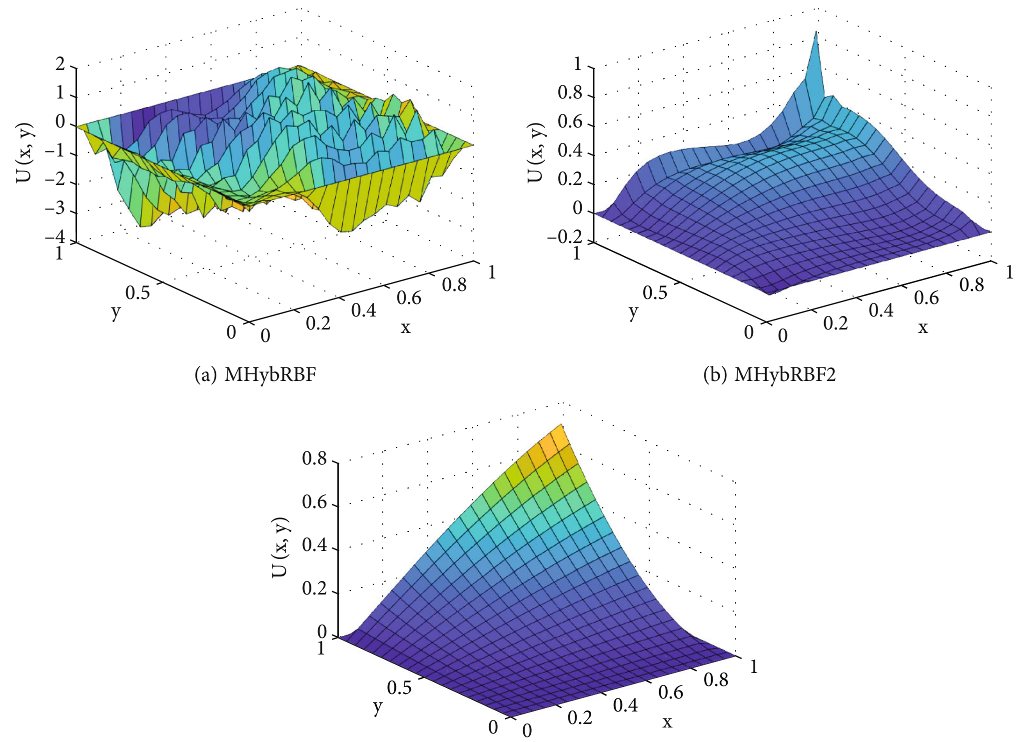


Figure 3: The comparison of RMSE of GA, CB, HybRBF, MHybRBF, HybRBF2, and MHybRBF2 with different ρ.

Table 3 shows the results of HybRBF2 and MHybRBF2 using the optimal shape parameter for different diffusion coefficients ρ, and Figure 2 depicts the comparison of the RMSE of the collocation method using HybRBF, MHybRBF, HybRBF2, and MHybRBF2. It can be found that choosing the optimal shape parameter of HybRBF2 and MHybRBF2 is helpful. For different values of ρ in the convection-diffusion problem, the errors obtained by both HybRBF2 and MHybRBF2 decrease, which indicates that HybRBF2 and MHybRBF2 can perform well and improve the solution of the convection-diffusion problem with different diffusion coefficients ρ including the convection-dominated problem.

Moreover, the comparison of the collocation method using all six RBFs is discussed, and the RMSE of these

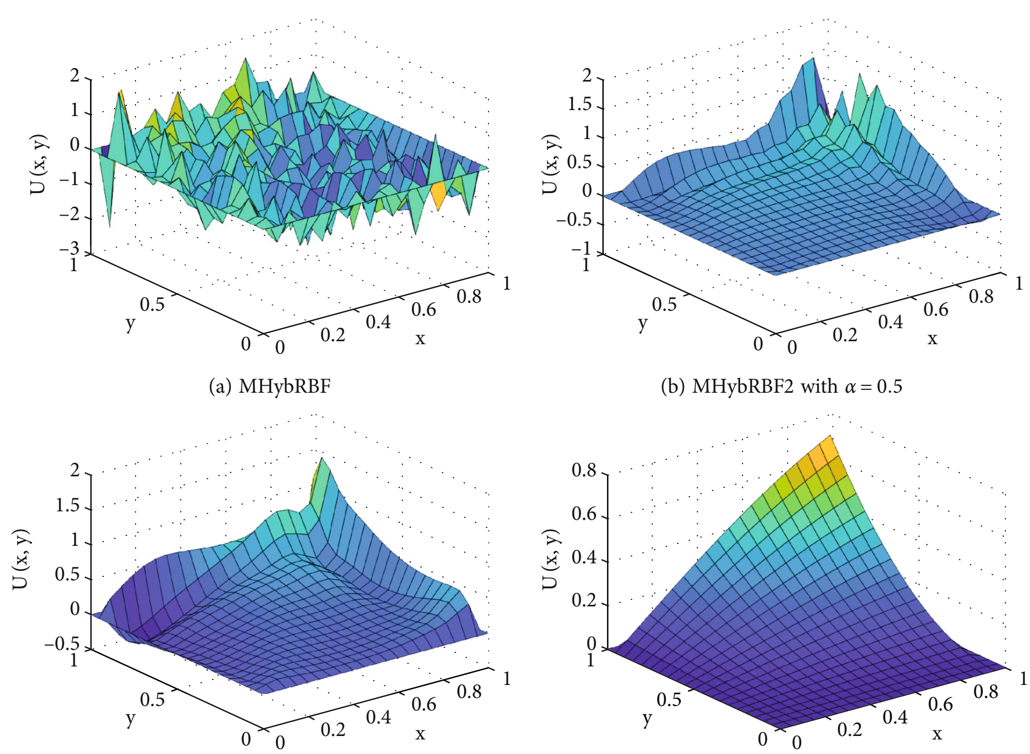


(a) MHybRBF

(b) MHybRBF2

(c) The analytical solution

Figure 4: The approximate solution of MHybRBF and MHybRBF2 for $\rho = 0.001$.



(a) MHybRBF

(b) MHybRBF2 with $\alpha = 0.5$

(c) MHybRBF2 with $\alpha = 0.95$

(d) The analytical solution

Figure 5: The comparison of the numerical results of MHybRBF with optimal α, ε and the analytical solution.

Table 4: The results of MHybRBF with optimal α, ε and different ρ.

ρ	α	ε	MHybRBF RMSE	MHybRBF MAXE	MHybRBF CN	Time
100	0.56	3.54	4.848361e-06	6.972640e-06	3.408575e+12	0.120996
10	0.95	3.56	1.791690e-05	3.549015e-05	5.803703e+10	0.120914
1	0.95	3.93	9.070883e-04	1.876517e-03	1.343570e+10	0.133980
0.5	0.95	8.93	2.556100e-04	8.988170e-04	5.582259e+09	0.124727
0.1	0.9	9.03	2.804490e-03	8.193785e-03	1.128277e+10	0.114545
0.01	0.98	33.70	3.355029e-02	3.797075e-01	1.780515e+10	0.125761
0.001	0.93	46.49	4.263947e-02	1.883747e-01	6.525949e+09	0.118694
0.0001	0.95	58.5	1.655461e-01	8.865616e-01	1.677765e+11	0.111952

methods is illustrated in Figure 3. It can be observed that the errors of the method using HybRBF2 and MHybRBF2 are quite smaller than those of the technique using other RBFs. It can also be found that, for different values of ρ, the proposed collocation method using MHybRBF, HybRBF2, and MHybRBF2 performs better than HybRBF. Furthermore, for $\rho = 0.001$, the approximate solutions obtained by applying MHybRBF and MHybRBF2 are compared and shown in Figure 4. As illustrated in Figure 4, MHybRBF2 provides the approximate solutions that agree with the exact solution, whereas MHybRBF still yields oscillatory solutions. In addition, in Tables 2 and 3, the CN of the matrix system is similar with or without the optimal shape parameter for the different values of ρ.

Although the proposed collocation method using MHybRBF yields slightly better results than those obtained by using HybRBF, both methods produce results that are not quite in good agreement with the exact solution, especially for a very small ρ. For example, when the proposed collocation method using MHybRBF is applied to $\rho = 0.0001$, the numerical results remain oscillatory and do not agree with those obtained by the exact solution, as shown in Figure 5(a). Therefore, for a very small diffusion coefficient ρ, finding the optimal weight α is discussed in depth to control the contribution of the hybrid RBF and improve the results of the MHybRBF. It can be seen in Table 4 that when the proposed approach is employed with the optimal values of ε and α, the errors obtained by this method decrease. This implies that the numerical results obtained by MHybRBF with the optimal weight α are improved and slightly agree with those obtained by the exact solution, as illustrated in Figures 5(b)–5(d).

In addition, in Table 4, the optimal weight α used for MHybRBF is found to be about 0.95 for solving the convection-diffusion problem with each diffusion coefficient ρ. On the other hand, the discovered optimal shape parameter ε shows an increasing trend as the diffusion coefficient ρ decreases. Also, the CN does not change significantly.

5. Conclusion

In this paper, a modified RBF collocation method using a hybrid RBF is proposed for the numerical solution of the convection-diffusion problem. A hybrid RBF proposed by Mishra et al. [36] is further utilized in conjunction with the RBF collocation method, and some promising results are obtained. The advantages of the Gaussian and cubic RBFs are extended to modify the hybrid RBF by adding the extra weight which controls the contribution of the Gaussian and cubic RBFs. It is worth noting that when using the modified hybrid RBF, the quality of the results is significantly improved and agrees with the exact solution. However, the performance of the proposed method will be reduced when the problem has a very small diffusion coefficient or is a convection-dominated problem. This means that the numerical solutions obtained using this method are still oscillatory for a small diffusion coefficient. To improve the accuracy of the modified RBF collocation method using the hybrid RBF, finding the optimal shape parameter of the Gaussian part and the optimal weight of a modified hybrid RBF is discussed and found through a numerical experiment with the minimum root mean square error. The numerical study demonstrates that choosing the optimal shape parameter produces better results, and the obtained results are acceptable and agree with the exact solution. Moreover, the proposed scheme can overcome the convection-dominated problem (the instability issue), which is a challenging problem in science and engineering. It can also be noted that the optimal weight of the hybrid RBF plays an important role in the numerical solution to the convection-diffusion problem, especially the convection-dominated. Moreover, the combination of the Gaussian RBF and cubic RBF works well. Finally, when stability is considered, the CN obtained by the modified RBF collocation method does not differ significantly in solving the convection-diffusion problem with different diffusion coefficients. However, the proposed approach yields a lower CN than using only the Gaussian RBF.

In future work, the proposed scheme will be extended to solve other types of PDEs, including the modeling of both time-independent and time-dependent PDEs. Another important aspect to investigate concerns the study of the nodal distributed model: irregular points and refinement points. Furthermore, the combination of other RBFs is worth investigating.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that she has no conflicts of interest.

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