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On Unitary N-Dilations for Tuples of Circulant Contractions and von Neumann's Inequality

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Abstract

We introduce the spectral mapping factorization of tuples of circulant matrices and its matrix version. We prove that every tuple of circulant contractions has a unitary N-dilation. We show that von Neumann's inequality holds for tuples of circulant contractions. We construct completely contractive homomorphisms over the algebra of complex polynomials defined on $\mathbb{D}^n$.

Keywords

Dilations, Polynomials, Matrices

1. Introduction

In 1953, Sz-Nagy [1] [2] showed that every single contraction on a Hilbert space has a unitary dilation. This is an interesting tool which can be used to prove the von Neumann inequality [3] [4] [5] [6] which states that for any contraction linear operator T on a Hilbert space the following inequality:

$$\|p(T)\| \leq \|p\|_{\infty},$$

holds for all complex polynomials $p(z)$ over the unit disk, where $\|p\|_{\infty}$ denotes the supremum norm of p over the unit disk. In 1963, Ando proved that every pair of commuting contractions has a simultaneous commuting dilation [7]. However, Varopoulos [8], Parrott [9] and Crabb-Davie [10] proved that this phenomenon fails for more than three commuting contractions. In 1978, Drury [11], in connection to his generalization of von Neumann's inequality, and then Arveson [12], in 1998, proved the standard commuting dilation for tuples of commuting contractions. The problem of determining if a tuple of commuting (or non-commuting) contractions admits a unitary dilation has been pursued by many authors. Over the years, several conditions that guarantee the existence of a unitary dilation for an n-tuple of commuting contractions have been studied [13]. For example: tuples of doubly commuting contractions have unitary N-dilations ($N \in \mathbb{N}$) acting on a finite dimensional space [14].

This result has many engineering applications [15].

In the present paper, we introduce the spectral mapping factorization of tuples of circulant matrices and its matrix version. Circulant matrices have many applications in graph theory, cryptography, physics, signal and image processing, probability, statistics, numerical analysis, algebraic coding theory and many other areas [16]. The well known results on unitary dilations of doubly commuting sets of contractions allow us to extend Sz-Nagy's Dilation Theorem and von Neumann's inequality to the setting of tuples of circulant contractions.

Theorem 1.1. *Let $N \in \mathbb{N}$ be a positive integer. Every tuple of circulant contractions has a unitary N-dilation.*

Theorem 1.2. *von Neumann's inequality holds for tuples of circulant contractions.*

The matrix version of the spectral mapping factorization of tuples of circulant matrices allows us to introduce a new family of completely contractive homomorphisms over the algebra of complex polynomials defined on $\mathbb{D}^n$.

2. Preliminaries

Throughout this paper H is a Hilbert space of finite dimension *n*. Let

$$D = \left[d_{i,j} \right]_{i,j=1}^{n} \in M_n\left(\mathbb{C}\right)$$

be a complex matrix. Denote by $D^* = \left[d_{j,i}^* \right]_{i,j=1}^{n}$.

2.1. Operator Norm

Definition 2.1. *Let $\mathbb{A}$ be a unital Banach algebra. We say that $a \in \mathbb{A}$ is invertible if there is an element $b \in \mathbb{A}$ such that $ab = ba = 1$. In this case b is unique and written a^{-1}. The set*

$$Inv\left(\mathbb{A}\right) = \left\{ a \in \mathbb{A} : \exists b \in \mathbb{A}, ab = ba = 1 \right\}$$

is a group under multiplication.

If a is an element of $\mathbb{A}$, the spectrum of a is defined as

$$\sigma\left(a\right) = \left\{ \lambda \in \mathbb{C} : a - \lambda.1 \notin Inv\left(\mathbb{A}\right) \right\},$$

and its spectral radius is defined to be

$$r\left(a\right) = \sup\left\{ \left|\lambda\right| : \lambda \in \sigma\left(a\right) \right\}.$$

Definition 2.2. *Let H be a Hilbert space and let $T \in \mathcal{B}\left(H\right)$ be a linear bounded operator on H. Then the operator norm of T denoted by $\left\|T\right\|$ is defined by*

$$\left\|T\right\| = \sup\left\{ \frac{\left\|T\left(x\right)\right\|}{\left\|x\right\|} : x \neq 0 \right\} = \sup\left\{ \left\|T\left(x\right)\right\| : \left\|x\right\| \leq 1 \right\} = \sup\left\{ \left\|T\left(x\right)\right\| : \left\|x\right\| = 1 \right\}.$$

If $\|T\| \le 1$, then the linear bounded operator T is called a contraction. In the case, H is a finite dimensional Hilbert space, then

$$\|T\| = \sqrt{r\left(T^*T\right)} = \sqrt{\sup\left\{|\lambda| : \lambda \in \sigma\left(T^*T\right)\right\}}.$$

2.2. Complex Polynomials

Let $\mathbb{D}^n = \left\{z \in \mathbb{C} : |z| \le 1\right\}$ be the unit poly disk and let

$$f\left(z_1,\cdots,z_n\right) = \sum_{(k_1,\cdots,k_n)\in S} \hat{f}\left(k_1,\cdots,k_n\right) z_1^{k_1}\cdots z_n^{k_n},$$

be a complex polynomial over $\mathbb{D}^n$. Then

$$\|f\|_\infty = \sup\left\{\left|f\left(z_1,\cdots,z_n\right)\right| : |z_1| = \cdots = |z_n| = 1\right\}.$$

Let $\mathcal{P}_n$ be the algebra of complex polynomials over $\mathbb{D}^n$. Given

$$f\left(z_1,\cdots,z_n\right) = \sum_{(k_1,\cdots,k_n)\in S} \hat{f}\left(k_1,\cdots,k_n\right) z_1^{k_1}\cdots z_n^{k_n},$$

a complex polynomial over $\mathbb{D}^n$, let us set

$$\|f\|_u = \sup\left\|f\left(T_1,\cdots,T_n\right)\right\|,$$

where the supremum is taken over the family of all *n*-tuples of contractions on all Hilbert spaces. It is easy to see that $\|f\|_u$ is finite, since it is bounded by the sum of the absolute values of the Fourier coefficients of *f*, and that this quantity defines a norm on the algebra $\mathcal{P}_n$ of complex polynomials over $\mathbb{D}^n$. For each polynomial *p* in $\mathcal{P}_n$, there is always an *n*-tuple of contractions where this supremum is achieved. Therefore, $\left(\mathcal{P}_n, \|.\|_\infty\right)$ and $\left(\mathcal{P}_n, \|.\|_u\right)$ are both two normed algebras.

Let $f_{i,j}\left(z_1,\cdots,z_n\right), i, j = 1,\cdots,m$, be complex polynomials in *n* variables over $\mathbb{D}^n$. Then

$$\left\|\left(f_{i,j}\right)\right\|_\infty = \sup\left\{\left\|\left(f_{i,j}\left(z_1,\cdots,z_n\right)\right)\right\|_{M_m} : |z_i| \le 1\right\}.$$

2.3. Unitary Dilation

Definition 2.3. *Let* $N \in \mathbb{N}$ *and let* $\left(T_1,\cdots,T_k\right)$ *be a tuple of commuting contractions on H. A unitary N-dilation for* $\left(T_1,\cdots,T_k\right)$ *is a k-tuple of commuting unitaries* $\left(U_1,\cdots,U_k\right)$ *acting on a space* $K \supseteq H$ *such that*

$$T_1^{n_1}\cdots T_k^{n_k} = P_H U_1^{n_1}\cdots U_k^{n_k}\Big|_H,$$

for all $n_1,\cdots,n_k$ *satisfying* $n_1 + \cdots + n_k \le N$.

Definition 2.4. *A finite set* $\left\{B_1,\cdots,B_n\right\}$ *of matrices is said to be doubly commuting if* $B_iB_j = B_jB_i$ *and* $B_i^*B_j = B_jB_i^*$, *for every* $i \ne j$.

The following results will enable us to prove our main results [13] [14].

Theorem 2.5. (*Sz-Nagy-Foias*). *Given k doubly commuting contractions* $T_1,\cdots,T_k \in B(H)$, *there is a Hilbert space K containing H and doubly commuting unitaries* $U_1,\cdots,U_k \in B(K)$ *so that*

$$T_1^{n_1} \cdots T_k^{n_k} = P_H U_1^{n_1} \cdots U_k^{n_k} \Big|_H ,$$

for all $n_1, \cdots, n_k \in \mathbb{Z}$.

Theorem 2.6. *Let* $(T_1, \cdots, T_k)$ *be a k-tuple of doubly commuting contractions on H. Then for every* $N \in \mathbb{N}$, *the k-tuple* $(T_1, \cdots, T_k)$ *has a unitary N-dilation that acts on a space of dimension* $(N+1)^k n$.

In finite dimensional, every tuple of commuting contractions which has a unitary N-dilation acting on a finite dimensional Hilbert space satisfies von Neumann's inequality [13].

Theorem 2.7. *Let* $N \in \mathbb{N}$, *and let* $(T_1, \cdots, T_k)$ *be a k-tuple of commuting contractions on H that has a unitary N-dilation acting on a finite dimensional Hilbert space K. Put m = dim K. Then there exist m points* $\left\{ w^i = \left(w_1^i, \cdots, w_k^i \right) \right\}_{i=1}^m$ *on the k-torus* $\mathbb{T}^k$ *such that for every polynomial* $f(z_1, \cdots, z_k)$ *of degree less than or equal to N,*

$$\left\| f(T_1, \cdots, T_k) \right\| \le \max \left\{ \left| f\left(w^i\right) \right| : i = 1, \cdots, m \right\}.$$

In particular,

$$\left\| f(T_1, \cdots, T_k) \right\| \le \sup \left\{ \left| f(z_1, \cdots, z_k) \right| : |z_i| = 1, i = 1, \cdots, k \right\} = \| f \|_\infty .$$

Now, let us turn our attention to a particular family of doubly commuting sets of matrices which have many applications in several areas such as graph theory, cryptography, physics, signal and image processing, probability, statistics, numerical analysis, algebraic coding theory and many other areas [16] [17].

2.4. Circulant Matrices

Let $\{a_0, a_1, \cdots, a_{m-1}\} \subset \mathbb{C}$ be a finite set of complex numbers, denote by $C(a_0, a_1, \cdots, a_{m-1})$ the following Toeplitz matrix:

$$C(a_0, a_1, \cdots, a_{m-1}) = \begin{pmatrix} a_0 & a_1 & \ddots & a_{m-1} \\ a_{m-1} & a_0 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ a_1 & \cdots & a_{m-1} & a_0 \end{pmatrix} \in M_n(\mathbb{C}).$$

This matrix is called a complex circulant matrix of order *m*. It is possible to write this matrix as a single variable matrix polynomial in *P*, where *P* is the cyclic permutation matrix given by

$$P = \begin{pmatrix} 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Indeed,

$$C(a_0, a_1, \cdots, a_{m-1}) = a_0 I_m + \sum_{k=1}^{m-1} a_k P^k .$$

The matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ a_9 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_8 & a_9 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_7 & a_8 & a_9 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_6 & a_7 & a_8 & a_9 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_5 & a_6 & a_7 & a_8 & a_9 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_1 \end{pmatrix}, a_i \in \mathbb{C},$$

is a 9 × 9-complex circulant matrix. It is well known that the set of $m \times m$-circulant matrices

$$\operatorname{Circ}(m) = \left\{ b_0 I_m + \sum_{k=1}^{m-1} b_k P^k : b_k \in \mathbb{C} \right\}$$

is a commutative algebra. Let $\varepsilon = \mathrm{e}^{\frac{2\pi i}{m}}$ be a primitive m-th root of unity. Let us denote by U the following matrix:

$$U = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 1 & \varepsilon & \cdots & \cdots & \varepsilon^{m-3} & \varepsilon^{m-2} & \varepsilon^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \varepsilon^{m-3} & \cdots & \cdots & \varepsilon^{(m-3)^2} & \varepsilon^{(m-3)(m-2)} & \varepsilon^{(m-1)(m-3)} \\ 1 & \varepsilon^{m-2} & \cdots & \cdots & \varepsilon^{(m-2)(m-3)} & \varepsilon^{(m-2)^2} & \varepsilon^{(m-1)(m-2)} \\ 1 & \varepsilon^{m-1} & \cdots & \cdots & \varepsilon^{(m-1)(m-3)} & \varepsilon^{(m-1)(m-2)} & \varepsilon^{(m-1)^2} \end{pmatrix}.$$

This matrix is called Vandermonde matrix. It is well known that this matrix has the following properties:

$$\det(U) = \frac{1}{m^{\frac{m}{2}}} \prod_{i,j=0}^{m-1} \left(\varepsilon^j - \varepsilon^i \right) \neq 0,$$

U is non-singular, unitary, $U^{-1} = \bar{U}^{\mathrm{T}}$, $U^{\mathrm{T}} = U$ and $U^{-1} = \bar{U} = U^*$. It is well known that all elements of Cir(m) are simultaneously diagonalised by the same unitary matrix U [18] [19] [20], that is, for A in Cir(m),

$$U^* A U = D_A$$

with D_A is a diagonal matrix with diagonal entries given by the ordered eigenvalues of A: $\lambda_1^A, \lambda_2^A, \cdots, \lambda_m^A$. The factorization $U^* A U = D_A$ is called the spectral factorization of A.

3. Proof of the Main Results

In this section, we introduce the spectral mapping factorization of tuples of circulant matrices and its matrix version. We prove our main two results.

Theorem 3.1. *Let A be a m × m-complex circulant matrix. Then*

$$\|A\| = \sup\{|\lambda| : \lambda \in \sigma(A)\} = r(A).$$

Proof. Let A be a $m \times m$-complex circulant matrix. The spectral factorization of the matrix A allows us to claim that

$$A = UD_AU^{-1}, D_A = \begin{pmatrix} \lambda_1^A & 0 & \ddots & 0 \\ 0 & \lambda_2^A & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_m^A \end{pmatrix}, \lambda_l^A \in \sigma(A).$$

It follows that

$$D_A = U^{-1}AU.$$

A simple calculation shows that

$$\|A\| \le \|U^{-1}\|\|D_A\|\|U\| \le \|D_A\|$$

and

$$\|D_A\| \le \|U\|\|A\|\|U^{-1}\| \le \|A\|.$$

Therefore,

$$\|A\| = \|D_A\| = \sup\{|\lambda| : \lambda \in \sigma(A)\} = r(A). \quad \square$$

Theorem 3.2. *Let A be a m × m-complex circulant matrix. Then*

$$\|f(A)\| = \sup\{|f(\lambda)| : \lambda \in \sigma(A)\} = r(f(A)), \forall f \in \mathcal{P}_1.$$

Proof. Let A be a $m \times m$-complex circulant matrix. We know that

$$A = UD_AU^{-1}, D_A = \begin{pmatrix} \lambda_1^A & 0 & \ddots & 0 \\ 0 & \lambda_2^A & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_m^A \end{pmatrix}, \lambda_l^A \in \sigma(A).$$

Then

$$f(D_A) = \begin{pmatrix} f(\lambda_1^A) & 0 & \ddots & 0 \\ 0 & f(\lambda_2^A) & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & f(\lambda_m^A) \end{pmatrix}, \lambda_i^A \in \sigma(A).$$

Therefore,

$$\|f(A)\| = \|U^{-1}f(D)U\| = \|f(D)\| = \sup\{|f(\lambda)| : \lambda \in \sigma(A)\} = r(f(A)). \quad \square$$

The spectral factorization of circulant matrices [17] allows us to establish the spectral mapping factorization of tuples of circulant matrices.

Theorem 3.3. *Let* $\mathbb{S} = \{A_1, A_2, \cdots, A_n\}$ *be a set of m × m-complex circulant matrices. Then*

$$f(A_1,\cdots,A_n)=U\begin{pmatrix} f(\lambda_1^{A_1},\cdots,\lambda_1^{A_n}) & 0 & \ddots & 0 \\ 0 & f(\lambda_2^{A_1},\cdots,\lambda_2^{A_n}) & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & f(\lambda_m^{A_1},\cdots,\lambda_m^{A_n}) \end{pmatrix}U^{-1},$$

$$\lambda_l^{A_i}\in\sigma(A_i),\forall f\in\mathcal{P}_n.$$

Proof. Let $\mathbb{S}=\{A_1,A_2,\cdots,A_n\}$ be a set of $m\times m$-complex circulant matrices. From the spectral factorization of every A_i, we can say that there exist a complex $m\times m$-unitary matrix U and $m\times m$-diagonal matrices $D_{A_i}, i=1,\cdots,n$ such that

$$A_i=UD_{A_i}U^{-1},D_{A_i}=\begin{pmatrix} \lambda_1^{A_i} & 0 & \ddots & 0 \\ 0 & \lambda_2^{A_i} & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_m^{A_i} \end{pmatrix},\lambda_l^{A_i}\in\sigma(A_i),l=1,\cdots,m.$$

A simple calculation shows that

$$f(A_1,\cdots,A_n)=Uf(D_{A_1},\cdots,D_{A_n})U^{-1},\forall f\in\mathcal{P}_n.$$

It is straightforward to see that

$$f(D_{A_1},\cdots,D_{A_n})=\begin{pmatrix} f(\lambda_1^{A_1},\cdots,\lambda_1^{A_n}) & 0 & \ddots & 0 \\ 0 & f(\lambda_2^{A_1},\cdots,\lambda_2^{A_n}) & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & f(\lambda_m^{A_1},\cdots,\lambda_m^{A_n}) \end{pmatrix},$$

$\forall f\in\mathcal{P}_n$. Therefore,

$$f(A_1,\cdots,A_n)=U\begin{pmatrix} f(\lambda_1^{A_1},\cdots,\lambda_1^{A_n}) & 0 & \ddots & 0 \\ 0 & f(\lambda_2^{A_1},\cdots,\lambda_2^{A_n}) & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & f(\lambda_m^{A_1},\cdots,\lambda_m^{A_n}) \end{pmatrix}U^{-1},$$

$$\lambda_l^{A_i}\in\sigma(A_i),\forall f\in\mathcal{P}_n.$$ □

Now, we are ready to establish the matrix version of the spectral mapping factorization of tuples of circulant matrices.

Theorem 3.4. *Let* $k\in\mathbb{N}$ *and let* $\mathbb{S}=\{A_1,A_2,\cdots,A_n\}$ *be a set of* $m\times m$*-complex circulant matrices and let* $\left[f_{i,j}(z_1,\cdots,z_n)\right]_{i,j=1}^k$ *be a* $k\times k$*-matrix with complex polynomials as entries. Let us denote by*

$$B=\left[f_{i,j}(A_1,\cdots,A_n)\right]_{i,j=1}^k$$

and

$$a_{i,j}=\begin{pmatrix} f_{i,j}\left(\lambda_1^{A_1},\cdots,\lambda_1^{A_n}\right) & 0 & \ddots & 0 \\ 0 & f_{i,j}\left(\lambda_2^{A_1},\cdots,\lambda_2^{A_n}\right) & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & f_{i,j}\left(\lambda_m^{A_1},\cdots,\lambda_m^{A_n}\right) \end{pmatrix},$$

$\lambda_l^{A_r}\in\sigma(A_r), i,j=1,\cdots,k$. Then

$$B=\begin{pmatrix} U & 0 & \ddots & 0 \\ 0 & U & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & U \end{pmatrix}\begin{pmatrix} a_{1,1} & a_{1,2} & \ddots & a_{1,k} \\ a_{2,1} & a_{2,2} & \ddots & a_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k,1} & \cdots & \cdots & a_{k,k} \end{pmatrix}\begin{pmatrix} U & 0 & \ddots & 0 \\ 0 & U & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & U \end{pmatrix}^{-1}.$$

Proof. Let $\mathbb{S}=\{A_1,A_2,\cdots,A_n\}$ be a set of $m\times m$-complex circulant matrices and let $\left[f_{i,j}(z_1,\cdots,z_n)\right]_{i,j=1}^{k}$ be a $k\times k$-matrix with complex polynomials as entries, $k\in\mathbb{N}$. Suppose that

$$B=\left[f_{i,j}(A_1,\cdots,A_n)\right]_{i,j=1}^{k}, f_{i,j}\in\mathcal{P}_n,$$

and

$$a_{i,j}=\begin{pmatrix} f_{i,j}\left(\lambda_1^{A_1},\cdots,\lambda_1^{A_n}\right) & 0 & \ddots & 0 \\ 0 & f_{i,j}\left(\lambda_2^{A_1},\cdots,\lambda_2^{A_n}\right) & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & f_{i,j}\left(\lambda_m^{A_1},\cdots,\lambda_m^{A_n}\right) \end{pmatrix},$$

$\lambda_l^{A_r}\in\sigma(A_r)$. Theorem 3.3 allows us to claim that

$$f_{i,j}(A_1,\cdots,A_n)=Ua_{i,j}U^{-1},$$

$f_{i,j}\in\mathcal{P}_n,\lambda_l^{A_r}\in\sigma(A_r)$. It follows that

$$B=\left[Ua_{i,j}U^{-1}\right]_{i,j=1}^{k}.$$

A simple calculation shows that

$$B=\begin{pmatrix} U & 0 & \ddots & 0 \\ 0 & U & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & U \end{pmatrix}\begin{pmatrix} a_{1,1} & a_{1,2} & \ddots & a_{1,k} \\ a_{2,1} & a_{2,2} & \ddots & a_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k,1} & \cdots & \cdots & a_{k,k} \end{pmatrix}\begin{pmatrix} U & 0 & \ddots & 0 \\ 0 & U & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & U \end{pmatrix}^{-1}. \quad \square$$

Proof of Theorem 1.1

We just need to show that every tuple of circulant contractions is a doubly commuting set of contractions. Let $k\in\mathbb{N}$ and let

$$\left(C_1\left(a_{0,1},a_{1,1},\cdots,a_{m-1,1}\right),\cdots,C_k\left(a_{0,k},a_{1,k},\cdots,a_{m-1,k}\right)\right)$$

be a k-tuple of circulant contractions. Assume that

$$P=\begin{pmatrix} 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}\in M_m\left(\mathbb{C}\right).$$

It is clear that

$$C_i\left(a_{0,i},a_{1,i},\cdots,a_{m-1,i}\right)=\begin{pmatrix} a_{0,i} & a_{1,i} & \ddots & a_{m-1,i} \\ a_{m-1,i} & a_{0,i} & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ a_{1,i} & \cdots & 0 & a_{0,i} \end{pmatrix}.$$

Also,

$$C_i\left(a_{0,i},a_{1,i},\cdots,a_{m-1,i}\right)=a_{0,i}I_m+\sum_{r=1}^{m-1}a_{r,i}P^r,1\le i\le k.$$

Let us show that the set

$$\left\{C_1\left(a_{0,1},a_{1,1},\cdots,a_{m-1,1}\right),\cdots,C_k\left(a_{0,k},a_{1,k},\cdots,a_{m-1,k}\right)\right\}$$

is a doubly commuting set of contractions. We already know that this set is commutative. Let us observe that

$$C_j\left(a_{0,j},a_{1,j},\cdots,a_{m-1,j}\right)^*=\overline{a_{0,j}}I_m+\sum_{r=1}^{m-1}\overline{a_{n-r,j}}P^r.$$

Therefore, the adjoint of a complex $m\times m$-circulant matrix is another $m\times m$-circulant matrix. The fact that the set of $m\times m$-circulant matrices is a commutative algebra implies that the matrices $C_i\left(a_{0,i},a_{1,i},\cdots,a_{m-1,i}\right)$ and $C_j\left(a_{0,j},a_{1,j},\cdots,a_{m-1,j}\right)^*, i\ne j$, commute. Therefore, the set

$$\left\{C_1\left(a_{0,1},a_{1,1},\cdots,a_{m-1,1}\right),\cdots,C_k\left(a_{0,k},a_{1,k},\cdots,a_{m-1,k}\right)\right\}$$

is a doubly commuting set of contractions. Theorem 2.5 and Theorem 2.6 allow us to claim that, for each $N\in\mathbb{N}$, the k-tuple of circulant contractions

$$\left\{C_1\left(a_{0,1},a_{1,1},\cdots,a_{m-1,1}\right),\cdots,C_k\left(a_{0,k},a_{1,k},\cdots,a_{m-1,k}\right)\right\}$$

has a unitary N-dilation acting on a finite dimensional Hilbert space. Finally, for each $N\in\mathbb{N}$, every tuple of circulant contractions has a unitary N-dilation. □

The above proof allows us to observe the following: every finite set of circulant matrices is a doubly commuting set of matrices. This enables us to prove our second main result.

First proof of Theorem 1.2

Let $\left(A_1,\cdots,A_n\right)$ be an n-tuple of circulant contractions of order m. The set $\left\{A_1,\cdots,A_n\right\}$ is a doubly commuting set of contractions. Theorem 2.7 allows us to claim that von Neumann's inequality holds for this set $\left\{A_1,\cdots,A_n\right\}$. Therefore,

$$\left\|f\left(A_1,\cdots,A_n\right)\right\|\le\left\|f\right\|_\infty,\forall f\in\mathcal{P}_n.$$ □

Second proof of Theorem 1.2

Let $\{A_1, A_2, \cdots, A_n\}$ be a set of $m \times m$-complex circulant contractions. Theorem 3.3 allows us to claim that

$$f(A_1,\cdots,A_n) = U\begin{pmatrix} f(\lambda_1^{A_1},\cdots,\lambda_1^{A_n}) & 0 & \ddots & 0 \\ 0 & f(\lambda_2^{A_1},\cdots,\lambda_2^{A_n}) & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & f(\lambda_m^{A_1},\cdots,\lambda_m^{A_n}) \end{pmatrix}U^{-1},$$

$\lambda_l^{A_i} \in \sigma(A_i), |\lambda_l^{A_i}| \le 1, \forall f \in \mathcal{P}_n$. It follows that

$$\|f(A_1,\cdots,A_n)\| = \left\| \begin{pmatrix} f(\lambda_1^{A_1},\cdots,\lambda_1^{A_n}) & 0 & \ddots & 0 \\ 0 & f(\lambda_2^{A_1},\cdots,\lambda_2^{A_n}) & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & f(\lambda_m^{A_1},\cdots,\lambda_m^{A_n}) \end{pmatrix} \right\|,$$

$\lambda_l^{A_i} \in \sigma(A_i)$, for all $f \in \mathcal{P}_n$. Therefore,

$$\|f(A_1,\cdots,A_n)\| \le \|f\|_\infty, \forall f \in \mathcal{P}_n. \quad \square$$

4. Application

In this section, we construct completely contractive homomorphisms over the algebra of complex polynomials defined on $\mathbb{D}^n$.

Theorem 4.1. *Let* $\mathbb{S} = \{A_1, A_2, \cdots, A_n\}$ *be a set of* $m \times m$*-complex circulant contractions. Then the map* $\phi^{\mathbb{S}} : \mathcal{P}_n \to \mathbb{M}_m$ *given by*

$$\phi^{\mathbb{S}}(f) = f(A_1,\cdots,A_n)$$

is a completely contractive homomorphism.

Proof. Let $\mathbb{S} = \{A_1, A_2, \cdots, A_n\}$ be a set of $m \times m$-complex circulant contractions. The spectral factorization of the matrix A_i allows us to claim that

$$A_i = UD_{A_i}U^{-1}, D_{A_i} = \begin{pmatrix} \lambda_1^{A_i} & 0 & \ddots & 0 \\ 0 & \lambda_2^{A_i} & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_m^{A_i} \end{pmatrix}, \lambda_l^{A_i} \in \sigma(A_i).$$

It follows that

$$D_{A_i} = U^{-1}A_iU.$$

A simple calculation shows that

$$\|A_i\| = \|D_{A_i}\| = \sup\{|\lambda| : \lambda \in \sigma(A_i)\}.$$

Finally, $|\lambda_l^{A_i}| \le 1, \lambda_l^{A_i} \in \sigma(A_i)$, since $\|A_i\| \le 1$. Suppose that $\phi^{\mathbb{S}} : \mathcal{P}_n \to \mathbb{M}_m$ is the map given by

$$\phi^{\mathbb{S}}(f) = f(A_1,\cdots,A_n).$$

It is well known that the map $\phi_{\mathbb{S}}$ is a homomorphism [14]. Also,

$$\left\|\phi^{\mathbb{S}}\right\| = \sup\left\{\frac{\left\|\phi^{\mathbb{S}}(f)\right\|}{\|f\|_{\infty}} : f \neq 0, f \in \mathcal{P}_n\right\}$$

$$\left\|\phi^{\mathbb{S}}\right\| = \sup\left\{\frac{\left\|f(A_1,\cdots,A_n)\right\|}{\|f\|_{\infty}} : f \neq 0, f \in \mathcal{P}_n\right\}.$$

First of all, let us show that the map $\phi_{\mathbb{S}}$ is a contractive map. Due to the fact that the elements of the set $\mathbb{S}=\{A_1,A_2,\cdots,A_n\}$ doubly commute implies that

$$\left\|f(A_1,\cdots,A_n)\right\| \leq \|f\|_{\infty}, \forall f \in \mathcal{P}_n.$$

Therefore,

$$\left\|\phi^{\mathbb{S}}\right\| = \sup\left\{\frac{\left\|f(A_1,\cdots,A_n)\right\|}{\|f\|_{\infty}} : f \neq 0, f \in \mathcal{P}_n\right\} \leq 1.$$

Let $k \in \mathbb{N}$ and define the map $\phi_k^{\mathbb{S}} : \mathbb{M}_k(\mathcal{P}_n) \to \mathbb{M}_k(\mathbb{M}_m)$ by setting

$$\phi_k^{\mathbb{S}}\left(\left[f_{i,j}\right]_{i,j=1}^k\right) = \left[f_{i,j}(A_1,\cdots,A_n)\right]_{i,j=1}^k$$

Let us show that the map $\phi_k^{\mathbb{S}}$ is contractive. Theorem 3.4 allows us to claim that if

$$B = \left[f_{i,j}(A_1,\cdots,A_n)\right]_{i,j=1}^k, f_{i,j} \in \mathcal{P}_n,$$

we can say that

$$B = \begin{pmatrix} U & 0 & \ddots & 0 \\ 0 & U & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & U \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & \ddots & a_{1,k} \\ a_{2,1} & a_{2,2} & \ddots & a_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k,1} & \cdots & \cdots & a_{k,k} \end{pmatrix} \begin{pmatrix} U & 0 & \ddots & 0 \\ 0 & U & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & U \end{pmatrix}^{-1}$$

with

$$a_{i,j} = \begin{pmatrix} f_{i,j}\left(\lambda_1^{A_1},\cdots,\lambda_1^{A_n}\right) & 0 & \ddots & 0 \\ 0 & f_{i,j}\left(\lambda_2^{A_1},\cdots,\lambda_2^{A_n}\right) & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & f_{i,j}\left(\lambda_m^{A_1},\cdots,\lambda_m^{A_n}\right) \end{pmatrix},$$

$\lambda_l^{A_r} \in \sigma(A_r)$. Recall that $\left|\lambda_l^{A_r}\right| \leq 1$, $l=1,\cdots,m$ and $r=1,\cdots,n$. It follows that

$$\left\|\left[f_{i,j}(A_1,\cdots,A_n)\right]_{i,j=1}^k\right\| = \left\|\begin{pmatrix} a_{1,1} & a_{1,2} & \ddots & a_{1,k} \\ a_{2,1} & a_{2,2} & \ddots & a_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k,1} & \cdots & \cdots & a_{k,k} \end{pmatrix}\right\| \leq \left\|\left[f_{i,j}\right]_{i,j=1}^k\right\|_{\infty}.$$

Finally,

$$\left\|\phi_k^{\mathbb{S}}\right\| = \sup\left\{\frac{\left\|\left[f_{i,j}(A_1,\cdots,A_n)\right]_{i,j=1}^k\right\|}{\left\|\left[f_{i,j}\right]_{i,j=1}^k\right\|_{\infty}} : \left[f_{i,j}\right]_{i,j=1}^k \in \mathbb{M}_k(\mathcal{P}_n)\right\} \leq 1, k \in \mathbb{N}. \qquad \square$$

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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