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Asymptotic Behavior of Solutions to Singular Quasilinear Dirichlet Problem with a Convection Term

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Abstract

In this paper, we study the boundary behavior of solution to the singular Dirichlet problem

$$\begin{cases} -\text{div}(|\nabla u|^{m-2}\nabla u) = b(x)g(u) + \lambda|\nabla u(x)|^{q(m-1)}, & x \in \Omega, \\ u > 0, \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where Ω is a bounded domain with smooth boundary in R^N, $\lambda \in R, m > 1, 0 < q \leq m/(m-1)$, $\lim_{s\to 0^+} g(s) = +\infty$, and $b \in C^\alpha(\overline{\Omega})$, which is non-negative on Ω and may be vanishing on the boundary, mainly, we investigate the exact asymptotic behavior of solution to the above problem.

Keywords: Dirichlet problem, quasilinear elliptic equation, asymptotic behavior, convection term.
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1 Introduction

In this paper, we plan to investigate the exact asymptotic behavior of solution to the following problem

$$\begin{cases} -\text{div}(|\nabla u|^{m-2}\nabla u) = b(x)g(u) + \lambda|\nabla u(x)|^{q(m-1)}, & x \in \Omega, \\ u > 0, \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $R^N (N \geq 1)$, $\lambda \in R, m > 1, 0 < q \leq m/(m-1)$, g satisfies

(g_1) $g \in C^1((0,\infty),(0,\infty)),\ g'(s) < 0$ for all $s > 0, \lim_{s\to 0^+} g(s) = +\infty$;

and b satisfies

(b_1) $b \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$, is non-negative in Ω and positive near the boundary $\partial\Omega$.

when $m = 2$, the problem (1.1) becomes

$$-\Delta u = b(x)g(u) + \lambda|\nabla u|^q,\ \ u > 0,\ x \in \Omega,\ \ u|_{\partial\Omega} = 0, \tag{1.2}$$

Problem (1.2) arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat condition in electrical materials(see [1-3]).

when $\lambda = 0$, problem (1.2) becomes

$$-\Delta u = b(x)g(u),\ \ u > 0,\ x \in \Omega,\ \ u|_{\partial\Omega} = 0, \tag{1.3}$$

problem was discussed in a number works (see[3-5]).

When $u|_{\partial\Omega} = 0$ becomes $u|_{\partial\Omega} = +\infty$, problem (1.1) becomes boundary blow-up elliptic problems

$$\begin{cases} -\text{div}(|\nabla u|^{m-2}\nabla u) = b(x)g(u) + \lambda|\nabla u(x)|^{q(m-1)}, & x \in \Omega, \\ u > 0, \quad x \in \Omega, \\ u|_{\partial\Omega} = +\infty, \end{cases} \tag{1.4}$$

When $m = 2$, the above problem becomes

$$-\Delta u = b(x)g(u) + \lambda|\nabla u(x)|^{q(m-1)},\ x \in \Omega,\ u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = +\infty, \tag{1.5}$$

many authors discussed the above problems[7-18].

In this paper, we consider the quasilinear elliptic problem (1.1). We modify the method developed by Zhang [6] and other authors' work, which showed the exact asymptotic behavior of solutions near the boundary to the quasilinear problem (1.1), extend and complement the results of [6] to a quasilinear elliptic problem (1.1).

Our main results are as follows:

Theorem 1.1. Let $\lambda \in R$, $0 < q \leq 1, 1 < m \leq 2$ (or $q \geq 1, m \geq 2$), b satisfies (b_1) g satisfies (g_1) and $g \in NRVZ_{-\gamma}$ with $\gamma > m-1$. Suppose that there exists a positive non-decreasing C^1-function $k \in NRVZ_{\sigma/2}$ with $\sigma \in [0, \frac{\gamma}{m-1} - 1)$ and a positive constant b_0 such that

(b_2) $\lim_{d(x)\to 0} \frac{b(x)}{k^m(d(x))} = b_0$,

then the solution $u_\lambda \in C(\overline{\Omega}) \cap C^2(\Omega)$ to problem (1.1) satisfies

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{\varphi_1(K(d(x)))} = \xi_0.$$

where $\xi_0^{-(\gamma+m-1)} = \frac{2(\gamma-(\sigma+1)(m-1))}{b_0(2+\sigma)(\gamma-m+1)}$ and $\varphi_1 \in C[0,a] \cap C^2(0,a]$ satisfies

$$\int_0^{\varphi_1(t)} \frac{ds}{\sqrt[m]{mG_2(s)}} = t,\ \ t \in [0,a] \ \text{ for small } \ a > 0, \tag{1.12}$$

$$K(t) = \int_0^t k(s)ds, \quad t \in [0, a]; \quad G_2(t) = \int_t^b g(s)ds, \quad t \in (0, b], \ b > 0. \tag{1.13}$$

Moreover, $\varphi_1 \in NRVZ_{2/(1+\gamma)}$ and there exists $y_2 \in C(0, a]$ with $\lim_{s\to 0^+} y_2(s) = 0$ such that $\varphi_1(t) = t^{2/(1+\gamma)} e^{\int_t^a \frac{y_2(s)}{s} ds}, \ t \in (0, a]$.

2 Preliminaries

In this section, we present some bases of the theory which come from Senta [19], Preliminaries in Resnick [20], Introductions and the appendix in Maric [21].

Definition 2.1. A positive measurable function f defined on $[a, +\infty)$, for some $a > 0$, is called **regularly varying at infinity** with index ρ, written as $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbf{R}$,

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{2.1}$$

In particular, when $\rho = 0, f$ is called **slowly varying at infinity**.

Definition 2.2. A positive measurable function f defined on $[a, +\infty)$, for some $a > 0$, is called **rapidly varying at infinity** if for each $p > 1$

$$\lim_{s\to\infty} \frac{f(s)}{s^p} = \infty. \tag{2.2}$$

Clearly, if $f \in RV_\rho$, then $L(s) := f(s)/s^\rho$ is slowly varying at infinity.

Proposition 2.1 (Uniform convergence theorem). If $f \in RV_\rho$,then (2.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$. Moreover, if $\rho < 0$, then uniform convergence holds on intervals of the form (a_1, ∞) with $a_1 > 0$; if $\rho > 0$, then uniform convergence holds on intervals $(a_1, \infty]$ provided f is bounded on $(a_1, \infty]$ for all $a_1 > 0$.

Proposition 2.2 (Representation theorem). A function L is slowly varying at infinity if and only if it may be written in the form

$$L(s) = \varphi(s) \exp(\int_{a_1}^s \frac{y(\tau)}{\tau} d\tau), \quad s \geq a_1, \tag{2.3}$$

for some $a_1 > a$, where the functions φ and y are measurable and for $s \to \infty, y(s) \to 0$, and$\varphi(s) \to c_0$,with $c_0 > 0$.

We call that

$$\hat{L}(s) = c_0 \exp(\int_{a_1}^s \frac{y(\tau)}{\tau} d\tau), \quad s \geq a_1, \tag{2.4}$$

is normalized slowly varying at infinity and

$$f(s) = c_0 s^\rho \hat{L}(s), \quad s \geq a_1, \tag{2.5}$$

is normalized regularly varying at infinity with index ρ (and written as $f \in NRV_\rho$).

Similarly, g is called normalized regularly varying at zero with index ρ ,written as $g \in NRVZ_\rho$ if $t \to g(1/t)$ belongs to NRV_ρ. A function $f \in RV_\rho$ belongs to NRV_ρ if and only if

$$f \in C^1[a_1, \infty), \text{ for some } a_1 > 0, \text{ and } \lim_{s\to\infty} \frac{sf'(s)}{f(s)} = \rho. \tag{2.6}$$

Proposition 2.3. If functions L, L_1 are slowly varying at infinity, then

(i) L^σ for every $\sigma \in \mathbf{R}$, $c_1 L + c_2 L_1$ $(c_1 \geq 0, c_2 \geq 0 \text{with} c_1 + c_2 > 0)$, $L \circ L_1$ (if $L_1(t) \to +\infty$ as $t \to +\infty$), are also slowly varying at infinity;

(ii) for every $\theta > 0$ and $t \to +\infty$,$t^\theta L(t) \to +\infty$ and $t^{-\theta} L(t) \to 0$;

(iii) for $\rho \in \mathbf{R}$ and $t \to +\infty$, $\frac{\ln(L(t))}{\ln t} \to 0$ and $\frac{\ln(t^\rho L(t))}{\ln t} \to \rho$.

Proposition 2.4. (Asymptotic behavior). If a function H is slowly varying at zero, then for $a > 0$ and $t \to 0^+$,

(i) $\int_a^t s^\beta H(s)ds \cong (\beta+1)^{-1} t^{1+\beta} H(t)$, for $\beta > -1$;

(ii) $\int_t^\infty s^\beta H(s)ds \cong (-\beta-1)^{-1} t^{1+\beta} H(t)$, for $\beta < -1$.

Corollary 2.1. If g satisfies (g_1) and $g \in NRVZ_{-\gamma}$ with $\gamma > 1$,then:

(i) $g(t) = t^{-\gamma} e^{\int_t^a \frac{y(s)}{s} ds},\ 0 < t < a,\ y \in C(0,a],\ \lim_{s\to 0^+} y(s) = 0$;

(ii) $\lim_{t\to 0^+} g(t) = +\infty = \lim_{t\to 0^+} G_2(t)$; $\lim_{t\to 0^+} \frac{G_2(t)}{g(t)} = 0 = \lim_{t\to 0^+} \frac{\sqrt[m]{G_2(t)}}{g(t)}$;

(iii) $\lim_{t\to 0^+} \frac{G_2(t)}{tg(t)} = \frac{1}{\gamma+1}$;$\lim_{t\to 0^+} \frac{tg'(t)}{g(t)} = -\gamma$.

Corollary 2.2. k in Theorem 1.1 has the following properties:

(i) $k(t) = t^{\sigma/2} e^{\int_t^a \frac{y_1(s)}{s} ds},\ 0 < t < a,\ y_1 \in C(0,a],\ \lim_{s\to 0^+} y_1(s) = 0$;

(ii) $\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0$; $\lim_{t\to 0^+} \frac{tk'(t)}{k(t)} = \sigma/2$; $\lim_{t\to 0^+} \frac{K(t)}{tk(t)} = 2/(2+\sigma)$;

(iii) $\lim_{t\to 0^+} \frac{k'(t)K(t)}{k^2(t)} = \lim_{t\to 0^+} \frac{tk'(t)}{k(t)} \lim_{t\to 0^+} \frac{K(t)}{tk(t)} = \sigma/(2+\sigma)$;

3 Proofs of the Main Results

First we give some preliminary considerations.

Lemma 3.1. Under the assumption in Theorem 1.1:

(i) $\varphi_1 \in NRVZ_{2/(1+\gamma)}$;

(ii) $(g \circ \varphi_1 \circ K)^{q-1} \cdot K^q \cdot k^{q-2} \in RVZ_\beta$ with $\beta = \frac{(2-q)\gamma + q(\sigma+1) - \sigma}{1+\gamma}$.

Proof. (i) We see by (2.6), the following Lemma 3.2(i) and Proposition 2.2(i) that $\varphi_1' \in NRVZ_{-(\gamma-1)/(1+\gamma)}$ and $\lim_{t\to 0^+} \frac{t\varphi_1'(t)}{\varphi_1(t)} = 2/(1+\gamma)$, Thus $\varphi_1 \in RVZ_{2/(1+\gamma)}$.

(ii) follows by (i) and Proposition 2.3.

Lemma 3.2. Let g, k and φ_1 be as in Theorem 1.1, then:

(i) $\lim_{t\to 0^+} \frac{\varphi_1'(t)}{t\varphi_1''(t)} = \frac{m-1-\gamma}{\gamma+1}$;

(ii) $\lim_{t\to 0^+} \frac{(\varphi_1'(t))^{(q-1)(m-1)+1}}{\varphi_1''(t)} = 0,\ q \in (0, m/(m-1)]$;

(iii) $\lim_{t\to 0^+} \frac{(k^{q(m-1)}(t)\varphi_1'(K(t)))^{(q-1)(m-1)+1}}{k^m(t)\varphi_1''(K(t))} = 0,\ q \in (0, m/(m-1)]$.

Proof. We see by (1.12) and a direct calculation that

$\varphi_1'(t) = \sqrt[m]{mG_2(\varphi_1(t))}, \quad -(\varphi_1'(t))^{m-2}\varphi_1''(t) = g(\varphi_1(t)), \quad 0 < t < a.$

(i) It follows by Corollary 2.1 and l'Hospital's rule that

$$\begin{aligned}\lim_{t\to 0^+} \frac{\varphi_1'(t)}{t\varphi_1''(t)} &= \lim_{t\to 0^+} \frac{(mG_2(\varphi_1(t)))^{1-\frac{1}{m}}}{-tg(\varphi_1(t))} = -\lim_{u\to 0^+} \frac{(mG_2(\varphi_1(t)))^{1-\frac{1}{m}}/g(u)}{\int_0^u \frac{ds}{\sqrt[m]{mG_2(s)}}} \\ &= -\lim_{u\to 0^+} \left[-(m-1) - \frac{mg'(u)G_2(u)}{g^2(u)} \right] \\ &= (m-1) + m \lim_{u\to 0^+} \frac{ug'(u)}{g(u)} \lim_{u\to 0^+} \frac{G_2(u)}{ug(u)} \\ &= \frac{m-1-\gamma}{\gamma+1}.\end{aligned}$$

(ii) $\lim_{t\to 0^+} \frac{(\varphi_1'(t))^2}{\varphi_1''(t)} = \lim_{t\to 0^+} \frac{(\varphi_1'(t))^2(\varphi_1'(t))^{m-2}}{-g(\varphi_1(t))} = \lim_{u\to 0^+} \frac{mG_2(u)}{-g(u)} = 0.$

Since $\lim_{t\to 0^+} \varphi_1'(t) = +\infty$, we have

$$\begin{aligned}\lim_{t\to 0^+} \frac{(\varphi_1'(t))^{(q-1)(m-1)+1}}{\varphi_1''(t)} &= \lim_{t\to 0^+} \frac{(\varphi_1'(t))^2}{\varphi_1''(t)} \lim_{t\to 0^+} (\varphi_1'(t))^{(q-1)(m-1)+1} \\ &= 0, \text{ for } 0 < q \le m/(m-1).\end{aligned}$$

(iii) We see by Lemma 3.1(ii) and Proposition 2.1(ii) that

$$\lim_{t\to 0^+} (g(\varphi_1(K(t))))^{q-1} K^q(t) k^{q-2}(t) = \lim_{t\to 0^+} t^\beta H(t) = 0,$$

where H is slowly varying at zero.

For $1 < m \le 2,\ 0 < q \le 1$, it follows that

$\lim_{t\to 0^+} \frac{k^{q(m-1)}(t)\left(\varphi_1'(K(t))\right)^{(q-1)(m-1)+1}}{k^m(t)\varphi_1''(K(t))}$
$= \lim_{t\to 0^+} \left(\frac{\varphi_1'(K(t))}{K(t)\varphi_1''(K(t))}\right)^{(q-1)(m-1)+1} \lim_{t\to 0^+} \left[(g(\varphi_1(K(t))))^{q-1}K^q(t)k^{q-2}(t)\right]^{m-1}$
$\lim_{t\to 0^+} \left(\frac{k(t)}{K(t)}\right)^{m-2} \left(\left[\varphi_1'(K(t))\right]^{(q-1)(m-1)}\right)^{-(m-2)}$
$= 0.$

For $m \geq 2,\ 1 < q \leq m/(m-1)$,

$\lim_{t\to 0^+} \frac{\left[\varphi_1'(K(t))\right]^{-(q-1)(m-1)}}{\frac{K(t)}{k(t)}}$
$= \lim_{t\to 0^+} \frac{-(m-1)(q-1)\left[\varphi_1'(K(t))\right]^{-(q-1)(m-1)-1}}{\frac{k^2(t)-k'(t)K(t)}{k^2(t)}}$
$= \lim_{t\to 0^+} \frac{-(m-1)(q-1)\left[\varphi_1'(K(t))\right]^{-(q-1)(m-1)-1}}{1-\frac{k'(t)K(t)}{k^2(t)}}$
$= 0;$

such that

$\lim_{t\to 0^+} \left(\frac{k(t)}{K(t)}\right)^{m-2} \left(\left[\varphi_1'(K(t))\right]^{(q-1)(m-1)}\right)^{-(m-2)}$
$= \left(\lim_{t\to 0^+} \frac{\left[\varphi_1'(K(t))\right]^{-(q-1)(m-1)}}{\frac{K(t)}{k(t)}}\right)^{m-2}$
$= 0.$

The proof is finished.

Proof of Theorem 1.1. Let $\xi_0^{-(\gamma+m-1)} = \tau_0/b_0$, where

$$\tau_0 = \frac{2[\gamma-(m-1)(\sigma-1)]}{(2+\sigma)(\gamma-m+1)} > 0, \qquad 1-\tau_0 = \frac{\sigma(\gamma+m-1)}{(2+\sigma)(\gamma-m+1)} > 0.$$

Fix $\varepsilon \in (0, \tau_0/4)$ and let

$$\xi_{1\varepsilon} = \left(\frac{b_0}{\tau_0 - 2\varepsilon}\right)^{1/(\gamma+m-1)}, \quad \xi_{2\varepsilon} = \left(\frac{b_0}{\tau_0 + 2\varepsilon}\right)^{1/(\gamma+m-1)}$$

It follows that

$$\left(\frac{2b_0}{3\tau_0}\right)^{1/(\gamma+m-1)} = C_1 < \xi_{2\varepsilon} < \xi_0 < \xi_{1\varepsilon} < C_2 = \left(\frac{2b_0}{\tau_0}\right)^{1/(\gamma+m-1)}.$$

Since $\partial\Omega \in C^2$, there exists a constant $\delta \in (0, \delta_0/2)$ which only depends on Ω such that
(i) $d(x) \in C^2(\overline{\Omega}_\delta)$ and $|\nabla d| \equiv 1$ on $\Omega_\delta = \{x \in \Omega : d(x < \delta)\}$.

By $(b_1), (b_2)$, corollary 2.2 and Lemma 3.2, we see that corresponding to ε, there is $\delta_\varepsilon \in (0, \delta)$ sufficiently small that:

(ii) For i=1,2,

$$\left|\frac{(m-1)k'(d(x))K(d(x))}{k^2(d(x))}\frac{\varphi_1'(s)}{s\varphi_1''(s)} - (\tau_0 - m + 1) + \frac{K(d(x))}{k(d(x))}\frac{\varphi_1'(s)}{s\varphi_1''(s)}\Delta d(x) + \frac{\lambda\xi_{i\varepsilon}^{(q-1)(m-1)}k^{q(m-1)}(d(x))}{k^m(d(x))}\frac{(\varphi_1'(K(d(x))))^{(q-1)(m-1)+1}}{\varphi_1''(K(d(x)))}\right| < \varepsilon, \quad \forall (x,s) \in \Omega_{\delta_\varepsilon} \times (0, \delta_\varepsilon)$$

(iii) For $x \in \Omega_{\delta_\varepsilon}$,

$$\frac{\xi_{2\varepsilon}^{m-1}k^m(d(x))g(\varphi_1(K(d(x))))}{g(\xi_{2\varepsilon}\varphi_1(K(d(x))))}(\tau_0+\varepsilon) < b(x) < \frac{\xi_{1\varepsilon}^{m-1}k^m(d(x))g(\varphi_1(K(d(x))))}{g(\xi_{1\varepsilon}\varphi_1(K(d(x))))}(\tau_0-\varepsilon),$$

Let $\bar{u}_\varepsilon = \xi_{1\varepsilon}\varphi_1(K(d(x))), \quad \underline{u}_\varepsilon = \xi_{2\varepsilon}\varphi_1(K(d(x))), \ x \in \Omega_{\delta_\varepsilon}$.
We see that for $x \in \Omega_{\delta_\varepsilon}$,

$$\begin{aligned}
&\text{div}(|\nabla \bar{u}_\varepsilon|^{m-2}\nabla \bar{u}_\varepsilon) + b(x)g(\bar{u}_\varepsilon(x)) + \lambda|\bar{u}_\varepsilon(x)|^{q(m-1)}\\
&= (m-1)\xi_{1\varepsilon}^{m-1}\Big(\varphi_1'(K(d(x)))\Big)^{m-2}\varphi_1''(K(d(x)))k^m(d(x)) + \xi_{1\varepsilon}^{m-1}\Big(\varphi_1'(K(d(x)))\Big)^{m-1}k^{m-1}(d(x))\cdot\\
&\quad \Delta d(x) + (m-1)\xi_{1\varepsilon}^{m-1}\Big(\varphi_1'(K(d(x)))\Big)^{m-1}k^{m-2}(d(x))k'(d(x)) + b(x)g(\xi_{1\varepsilon}\varphi_1(K(d(x))))\\
&\quad +\lambda\xi_{1\varepsilon}^{q(m-1)}\Big(\varphi_1'(K(d(x)))\Big)^{q(m-1)}k^{q(m-1)}(d(x))\\
&= \xi_{1\varepsilon}^{m-1}g(\varphi_1(K(d(x))))k^m(d(x))\bigg\{\frac{b(x)g(\xi_{1\varepsilon}\varphi_1(K(d(x))))}{\xi_{1\varepsilon}^{m-1}k^m(d(x))g(\varphi_1(K(d(x))))} - \tau_0\\
&\quad -\bigg(\frac{(m-1)k'(d(x))K(d(x))}{k^2(d(x))}\frac{\varphi_1'(K(d(x)))}{K(d(x))\varphi_1''(K(d(x)))} - (\tau_0 - m + 1)\bigg)\\
&\quad -\frac{K(d(x))}{k(d(x))}\frac{\varphi_1'(K(d(x)))}{K(d(x))\varphi_1''(K(d(x)))}\triangle d(x) - \frac{\lambda\xi_{i\varepsilon}^{(q-1)(m-1)}k^{q(m-1)}(d(x))}{k^m(d(x))}\frac{(\varphi_1'(K(d(x))))^{(q-1)(m-1)+1}}{\varphi_1''(K(d(x)))}\bigg\}\\
&\le 0;
\end{aligned}$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of problem (1.1) in $\Omega_{\delta_\varepsilon}$.
and

$$\begin{aligned}
&\text{div}(|\nabla \underline{u}_\varepsilon|^{m-2}\nabla \underline{u}_\varepsilon) + b(x)g(\underline{u}_\varepsilon(x)) + \lambda|\underline{u}_\varepsilon(x)|^{q(m-1)}\\
&= (m-1)\xi_{2\varepsilon}^{m-1}\Big(\varphi_1'(K(d(x)))\Big)^{m-2}\varphi_2''(K(d(x)))k^m(d(x)) + \xi_{2\varepsilon}^{m-1}\Big(\varphi_1'(K(d(x)))\Big)^{m-1}k^{m-1}(d(x))\cdot\\
&\quad \Delta d(x) + (m-1)\xi_{2\varepsilon}^{m-1}\Big(\varphi_1'(K(d(x)))\Big)^{m-1}k^{m-2}(d(x))k'(d(x)) + b(x)g(\xi_{2\varepsilon}\varphi_1(K(d(x))))\\
&\quad +\lambda\xi_{2\varepsilon}^{q(m-1)}\Big(\varphi_1'(K(d(x)))\Big)^{q(m-1)}k^{q(m-1)}(d(x))\\
&= \xi_{2\varepsilon}^{m-1}g(\varphi_1(K(d(x))))k^m(d(x))\bigg\{\frac{b(x)g(\xi_{2\varepsilon}\varphi_1(K(d(x))))}{\xi_{2\varepsilon}^{m-1}k^m(d(x))g(\varphi_1(K(d(x))))} - \tau_0\\
&\quad -\bigg(\frac{(m-1)k'(d(x))K(d(x))}{k^2(d(x))}\frac{\varphi_1'(K(d(x)))}{K(d(x))\varphi_1''(K(d(x)))} - (\tau_0 - m + 1)\bigg)\\
&\quad -\frac{K(d(x))}{k(d(x))}\frac{\varphi_1'(K(d(x)))}{K(d(x))\varphi_1''(K(d(x)))}\triangle d(x) - \frac{\lambda\xi_{2\varepsilon}^{(q-1)(m-1)}k^{q(m-1)}(d(x))}{k^m(d(x))}\frac{(\varphi_1'(K(d(x))))^{(q-1)(m-1)+1}}{\varphi_1''(K(d(x)))}\bigg\}\\
&\ge 0;
\end{aligned}$$

i.e., $\underline{u}_\varepsilon$ is a subsolution of of problem (1.1) in $\Omega_{\delta_\varepsilon}$.

Let $u_\lambda \in C(\overline{\Omega}) \cap C^{2+\alpha}(\Omega)$ be the solution to problem (1.1). We assert $\underline{u}_\varepsilon(x) \le u_\lambda(x) \le \bar{u}_\varepsilon(x)$, $\forall x \in \Omega_{\delta_\varepsilon}$.

In fact, denote $\Omega_{\delta_\varepsilon} = \Omega_{\delta_+} \cup \Omega_{\delta_-}$, where $\Omega_{\delta_+} = \{x \in \Omega_{\delta_\varepsilon} : \underline{u}_\varepsilon(x) \le u_\lambda(x)\}$ and $\Omega_{\delta_-} = \{x \in \Omega_{\delta_\varepsilon} : \underline{u}_\varepsilon(x) > u_\lambda(x)\}$.

We need to show $\Omega_{\delta_-} = \emptyset$. Assume the contrary, we see that there exists $x_0 \in \Omega_{\delta_-}$ (note that $\underline{u}_\varepsilon(x) = u_\lambda(x), \forall x \in \partial\Omega_{\delta_-}$) such that

$$0 < \underline{u}_\varepsilon(x_0) - u_\lambda(x_0) = \max_{x \in \overline{\Omega}_{\delta_-}} (\underline{u}_\varepsilon(x) - u_\lambda(x))$$

and

$$\nabla \underline{u}_\varepsilon(x_0) = \nabla u_\lambda(x_0), \quad \triangle(\underline{u}_\varepsilon - u_\lambda)(x_0) \le 0.$$

On the other hand, we see by (b_1) and (g_1) that

$$-\triangle(\underline{u}_\varepsilon - u_\lambda)(x_0) = b(x_0)(g(\underline{u}_\varepsilon(x_0)) - g(u_\lambda(x_0))) < 0,$$

which is a contradiction. Hence $\Omega_{\delta_-} = \emptyset$, i.e.,$\underline{u}_\varepsilon(x) \le u_\lambda(x), \ \forall x \in \Omega_{\delta_\varepsilon}$. In the same way, we can see that $\bar{u}_\varepsilon(x) \ge u_\lambda(x), \ \forall x \in \Omega_{\delta_\varepsilon}$.

It follows that

$$\xi_{2\varepsilon} \leq \lim_{d(x)\to 0} \inf \frac{u_\lambda(x)}{\varphi_1(K(d(x)))} \leq \lim_{d(x)\to 0} \sup \frac{u_\lambda(x)}{\varphi_1(K(d(x)))} \leq \xi_{1\varepsilon}.$$

Thus let $\varepsilon \to 0$, we see that

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{\varphi_1(K(d(x)))} = \xi_0.$$

The last part of the proof follows from Lemma 3.1(i).

4 Conclusion

The boundary value quasilinear differential equation systems (1.1) are mathematical models occurring in the studies of the p-Laplace equation, generalized reaction-diffusion theory, non-Newtonian fluid theory, and the turbulent flow of a gas in porous medium. In the non-Newtonian fluid theory, the quantity m is characteristic of the medium. Media with $m > 2$ are call dilatant fluids and those with $m < 2$ are called pseudoplastics. If $m = 2$, they are Newtonain fluids. When $m \neq 2$, the problem becomes more complicated since certain nice properties in herent to the case $m = 2$ seem to be lost or at least difficult to verify. The main differences between $m = 2$ and $m \neq 2$ can be founded in [14,22]. When $m = 2$, it is well known that all the positive solutions in $C^2(B_R)$ of the problem

$$\begin{cases} \triangle u + f(u) = 0 \text{ in } B_R \\ u(x) = 0 \text{ on } \partial B_R \end{cases}$$

are radially symmetric solutions for very general f(see [23]). Unfortunately, this result does not apply to the case $m \neq 2$. Kichenassary and Smoller showed that there exist many positive nonradial solutions of the above problem for some f (see [24]). The major stumbling block in the case of $m \neq 2$ is that certain nice features inherent to the case $m = 2$ seem to be lost or at least difficult to verify.

In this paper, we have two main findings as follows:

The first one is the asymptotic behavior of solutions to the following singular quasilinear Dirichlet problem

$$\begin{cases} -\text{div}(|\nabla u|^{m-2}\nabla u) = b(x)g(u) + \lambda|\nabla u(x)|^{q(m-1)}, \qquad x \in \Omega, \\ u > 0, \qquad x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

which is

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{\varphi_1(K(d(x)))} = \xi_0.$$

The second one is the corresponding proof method of the asymptotic behavior, which is the super-subersolutin method, the most critical point is the construction of the supersolution and subersolution.

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Competing Interests

The authors declare that no competing interests exist.

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