



# Pseudo-Lindley Family of Distributions and its Properties

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This work was carried out in collaboration between both authors. Author UUU designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. Author EEN managed the analyses of the study and managed the literature searches. Both authors read and approved the final manuscript.

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## Abstract

In this study, we proposed a family of distribution called the Pseudo Lindley family of distributions. The limiting behaviors of the density and hazard rate function of the new family are examined. Statistical properties of the proposed family of distributions derived include quantile function, moments, order statistics, and Renyi's entropy. The maximum likelihood method was employed in obtaining the parameter estimates of the Pseudo Lindley family of distribution. Bivariate extension of the proposed family is discussed. Some special members of the family are obtained. The shape of the density function of special members could be unimodal, bathtub shaped, increasing and decreasing.

**Keywords:** Pseudo lindley;  $T-R\{Y\}$  family; maximum likelihood; entropy; order statistics, bivariate distribution.

## 1 Introduction

The Probability Distribution function (pdf) of two parameter Pseudo-Lindley distribution with scale and shape parameter  $\theta$ ,  $\beta$  respectively is given by.

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$$f(x) = \frac{\theta(\beta - 1 + \theta x)e^{-\theta x}}{\beta}, x, \theta > 0, \beta \geq 1 \tag{1.1}$$

The cumulative distribution function (cdf) corresponding to (1.1) is

$$F(x) = 1 - \frac{(\beta + \theta x)e^{-\theta x}}{\beta} \tag{1.2}$$

If  $\beta = \theta + 1$ ,  $X$  now follows a Lindley distribution.

The Pseudo – Lindley distribution was proposed by Zeghdoudi and Nedjar [1]. The properties of Pseudo – Lindley distribution were studied by Zeghdoudi and Nedjar [2]. In the last decade many generalizations of different distributions and their applications have been proposed in literature to take care of data sets that do not follow known standard distribution. To make a distribution more flexible, it is usually generalized. This work we will make attempt to generate a family of distribution called Pseudo-Lindley family of distribution using the T-R {Y} framework by Alzaatreh et al [3]. Following the technique proposed by Aljarrah et al [4], Alzaatreh et al [4] defined the cdf of a random variable  $X$  as

$$F_X(x) = \int_a^{Q_Y(F_R(x;\xi))} f_T(t) dt = P[T \leq Q_Y(F_R(x;\xi))] = F_T(Q_Y(F_R(x;\xi))) \tag{1.3}$$

where  $F_X(x)$  is the cdf of the new family,  $f_T(x)$  is the pdf of the generator,  $F_R(x;\xi)$  is the cdf of the baseline distribution.  $Q_Y(F_R(x;\xi))$  is the quartile function of random variable  $Y$  expressed as function of  $F_R(x;\xi)$  the cdf of the base line distribution which depends on  $(r \times 1)$  parameter vector  $\xi$ . In this work, our generator  $f_T(x)$  is the pdf of a Pseudo – Lindley distribution.  $Q_Y(\cdot)$  is the quantile function of a standard exponential distribution.

We define the Pseudo-Lindley–G family by integrating the Pseudo-Lindley cdf in (1.1) as follows

$$F_X(x; \theta, \beta, \xi) = \int_0^{-\log[1-F_R(x;\xi)]} \frac{\theta(\beta - 1 + \theta t)e^{-\theta t}}{\beta} dt \tag{1.4}$$

From (1.4) we have

$$F_X(x; \theta, \beta, \xi) = 1 - \frac{(\beta - \theta \log(1 - F_R(x;\xi)))(1 - F_R(x;\xi))^\theta}{\beta} \tag{1.5}$$

The pdf of the Pseudo – Lindley family is obtained by differentiating (1.5) with respect to  $x$ .

$$f_X(x; \theta, \beta, \xi) = \frac{\theta}{\beta} f_R(x; \xi) [(\beta - 1) - \theta \log(1 - F_R(x; \xi))] [1 - F_R(x; \xi)]^{\theta-1} \tag{1.6}$$

where  $f_R(x; \xi)$  is pdf of the baseline distribution. Hereafter a random variable  $X$  with density function (1.6) is denoted by  $X \square PsL-R\{exponential\}$ . The survival function corresponding (1.5) is

$$S_X(x; \theta, \beta, \xi) = \frac{(\beta - \theta \log(1 - F_R(x; \xi)))(1 - F_R(x; \xi))^\theta}{\beta} \tag{1.7}$$

While the hazard rate function (*hrf*)  $X$  is given by

$$h_X(x; \theta, \beta, \xi) = \frac{\theta f_R(x; \xi) [(\beta - 1) - \theta \log(1 - F_R(x; \xi))]}{[\beta - \theta \log(1 - F_R(x; \xi))] [1 - F_R(x; \xi)]} \tag{1.8}$$

The rest of the paper is organized as follows. In Section 2 we present the statistical properties of the new family of distribution such as limiting behaviors, quantile function and moments while Section 3 is centered on the entropy of the proposed family of distribution. The distribution of order statistics and estimation of parameters of this new family are presented in Section 4 and 5 respectively. A bivariate extension of the generated family is given in Section 6. New generated distributions from the proposed family are given in Section 7 and Section 8 concludes the article.

## 2 Statistical Properties

The statistical properties of the  $PsL - R\{exponential\}$  family are studied in this section.

### 2.1 Limiting behaviors

We now examine behaviors of the cdf, pdf,  $S_X(x)$  and  $h_X(x)$ , as  $x \rightarrow 0$  and as  $x \rightarrow \infty$

**Proposition 1:** The limiting behaviors of (1.5), (1.6), (1.7) and (1.8) as  $x \rightarrow 0$  are given by

$$F_X(x; \theta, \beta, \xi) \rightarrow 0 \text{ as } x \rightarrow 0$$

$$f_X(x; \theta, \beta, \xi) \rightarrow \frac{\theta}{\beta} (\beta - 1) f_R(x; \xi) \text{ as } x \rightarrow 0$$

$$S_X(x; \theta, \beta, \xi) \rightarrow 1 \text{ as } x \rightarrow 0$$

$$h_X(x; \theta, \beta, \xi) \rightarrow \frac{\theta}{\beta} (\beta - 1) f_R(x; \xi) \text{ as } x \rightarrow 0$$

**Proposition 2:** The limiting behaviors of (1.5), (1.6), (1.7) and (1.8) as  $x \rightarrow \infty$  are given

$$F_X(x; \theta, \beta, \xi) \rightarrow 1 \text{ as } x \rightarrow \infty$$

$$f_X(x; \theta, \beta, \xi) \rightarrow 0 \text{ as } x \rightarrow \infty$$

$$S_X(x; \theta, \beta, \xi) \rightarrow 0 \text{ as } x \rightarrow \infty$$

$$h_x(x; \theta, \beta, \xi) \rightarrow 0 \text{ as } x \rightarrow \infty$$

### 2.2 Shapes

The shapes of the density and  $h_x(x)$  can be described analytically. The critical points of the density function of  $PsL-R\{exponential\}$  family are the roots of the equation.

$$\frac{d \log f_x(x; \xi)}{dx} = \frac{f'_R(x; \xi)}{f_R(x; \xi)} + \frac{\theta f_R(x; \xi)}{[(\beta - 1) - \theta \log(1 - F_R(x; \xi))]} - \frac{(\theta - 1) f_R(x; \xi)}{(1 - F_R(x; \xi))} \tag{2.1}$$

There may be more than one root to (2.1). If  $x = x_0$  is a root of (2.1), then it corresponds to a local maximum, local minimum or a point of inflexion depending on whether  $\psi(x_0) < 0, \psi(x_0) > 0$  or  $\psi(x_0) = 0$  where

$$\psi(x) = \frac{d^2 \log[f(x; \xi)]}{dx^2}$$

The critical points of the hazard rate function are obtained from the equation

$$\begin{aligned} \frac{d \log h_x(x; \xi)}{dx} = & \frac{f'_R(x; \xi)}{f_R(x; \xi)} + \frac{\theta f_R(x; \xi)}{[(\beta - 1) - \theta \log(1 - F_R(x; \xi))](1 - F_R(x; \xi))} \\ & - \frac{\theta f_R(x; \xi)}{[\beta - \theta \log(1 - F_R(x; \xi))](1 - F_R(x; \xi))} - \frac{f_R(x; \xi)}{(1 - F_R(x; \xi))} \end{aligned} \tag{2.2}$$

The roots of (2.2) may be more than one. If  $x = x_0$  is a root of (2.2), then it corresponds to a local maximum, local minimum or a point of inflexion depending on whether  $\zeta(x_0) < 0, \zeta(x_0) > 0$  or  $\zeta(x_0) = 0$ , where

$$\zeta(x) = \frac{d^2 \log(h_x(x; \xi))}{dx^2}$$

### 2.3 Useful expansions

The following expansions are considered useful in obtaining the moments, order statistics and entropy function of the  $PsL-R\{exponential\}$  family of distributions

$$(1 - z)^t = \sum_{i=0}^{\infty} (-1)^i \binom{t}{i} z^i \quad |z| < 1 \tag{2.3}$$

$$\log(1 - z) = - \sum_{i=0}^{\infty} \frac{z^{i+1}}{i+1} \quad |z| < 1 \tag{2.4}$$

$$\log(1 + z) = - \sum_{i=0}^{\infty} \frac{(-1)^{i+1} z^{i+1}}{i+1} \quad |z| < 1 \tag{2.5}$$

The power series raised to any positive integer expansion by Grandshteyn and Ryzhik [5].

$$\left(\sum_{i=0}^{\infty} a_i u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i} u^i \tag{2.6}$$

where coefficients  $c_{n,i}$   $i = 1, 2 \dots$  for  $c_{n,0} = a_0^n$  are easily obtained from the recurrence equation

$$c_{n,i} = \frac{1}{i a_0} \sum_{m=1}^i [m(n+1) - i] a_m c_{n,i-m}$$

### 2.4 Other representations

If  $X$  follows a  $PsL-R\{exponential\}$  family of distribution, a double mixture form of the  $PsL-R\{exponential\}$  family is obtained using (2.3) and (2.4)

$$f_X(x; \theta, \beta, \xi) = \frac{\theta}{\beta} (\beta - 1) f_R(x; \xi) \sum_{i=0}^{\infty} w_i [F_R(x; \xi)]^i + \frac{\theta^2}{\beta} f_R(x; \xi) \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} w_{ik} [F_R(x; \xi)]^{i+k+1} \tag{2.7}$$

where  $w_i = (-1)^i \binom{\theta-1}{i}$ ,  $w_{i,k} = (-1)^i \binom{\theta-1}{i} \frac{1}{k+1}$

In literature, if  $F(x)$  is any arbitrary cdf of a random variable, then for  $\delta > 0$   $G(x) = [F(x)]^\delta$  and  $g(x) = \delta f(x) [F(x)]^{\delta-1}$  are the cdf and pdf of exponentiated-G distribution introduced by Mudholkar and Stravstava [6]. The concept of exponentiated-G distribution is used to derive a very useful linear representation of the pseudo  $PsL-R\{exponential\}$  distribution.

$$f_X(x; \theta, \beta, \xi) = \sum_{i=0}^{\infty} w_i^* (i+1) f_R(x; \xi) [F_R(x; \xi)]^{(i+1)-1} + \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} w_{i,k}^* (i+k+2) f_R(x; \xi) [F_R(x; \xi)]^{(k+i+2)-1} \tag{2.8}$$

where  $w_i^* = \frac{\theta}{\beta} (\beta - 1) w_i$ ;  $w_{i,k}^* = \frac{\theta^2}{\beta} \frac{1}{(k+i+2)} w_{i,k}$

Hence 2.8 can be written as

$$f_X(x; \theta, \beta, \xi) = \sum_{i=0}^{\infty} w_i^* h_Y(y; i+1) + \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} w_{i,k}^* h_Y(y; (i+k+2)) \tag{2.9}$$

Where  $h_Y(\cdot)$  follows the exponentiated-G distribution, (2.9) is the major result in this section. The statistical properties of  $PsL-R\{exponential\}$  family can now be easily derived using the properties of exponentiated- G  $h_Y(\cdot)$  which has been studied widely in literature.

### 2.5 Quantile function

The quantile function of a random variable  $X \sim PsL-R\{Exponential\}$  distribution defined by  $Q(p) = X$  where  $0 < p < 1$  can be obtained by inverting (1.5) and its is given by

$$Q(p) = Q_R \left[ 1 - e^{\left( \frac{\beta + 1}{\theta} W(-\beta(1-p)e^{-\beta}) \right)} \right] \tag{2.10}$$

where  $Q_R(\cdot)$  is the quantile function of the baseline distribution,  $W(\cdot)$  is the negative branch of Lambert W function. The first three quartiles of the proposed family can be obtained by substituting for  $p = \frac{1}{4}, \frac{1}{2}$  and  $\frac{3}{4}$  in (2.10)

$$\begin{aligned} Q_1 &= Q\left(\frac{1}{4}\right) = Q_R \left[ 1 - e^{\left( \frac{\beta + 1}{\theta} W\left(-\beta\left(\frac{3}{4}\right)e^{-\beta}\right) \right)} \right] \\ Median(Md) &= Q_2 = Q\left(\frac{1}{2}\right) = Q_R \left[ 1 - e^{\left( \frac{\beta + 1}{\theta} W\left(-\beta\left(\frac{1}{2}\right)e^{-\beta}\right) \right)} \right] \\ Q_3 &= Q\left(\frac{3}{4}\right) = Q_R \left[ 1 - e^{\left( \frac{\beta + 1}{\theta} W\left(-\beta\left(\frac{1}{4}\right)e^{-\beta}\right) \right)} \right] \end{aligned} \tag{2.11}$$

(2.10) can be used for simulating the  $PsL-R\{exponential\}$  random variable. If  $p$  has a uniform  $U(0,1)$  distribution, then  $X$  given by  $X = Q_R \left[ 1 - e^{\left( \frac{\beta + 1}{\theta} W(-\beta(1-p)e^{-\beta}) \right)} \right]$  has the density in (1.6)

Skewness and kurtosis are used to measure the degree of long tail and the degree of tail heaviness respectively of a distribution. Quantile based measures of skewness and kurtosis are respectively calculated using the relationships of Galton [7] and Moor [8]. These measures of skewness and kurtosis exist for distributions without moments and are less sensitive to outliers Alizadeh et al [9].Using the quantile function in (2.10), the Galton’s skewness and Moor’s kurtosis of the proposed family are given by (2.12) and (2.13).

$$S = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{2.12}$$

$$K = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \tag{2.13}$$

If  $S > 0$  (or  $S < 0$ ) we say that the distribution is right or left skewed. If  $S = 0$ , then the distribution is symmetric. For the kurtosis  $K$ , as  $K$  increases the tail of the distribution becomes heavier.

### 2.6 Moment

Let  $Y_i$  and  $Y_{i+k}$  be random variables with exponentiated-G distribution with power parameters  $(i + 1)$  and  $(i + k + 2)$  respectively. The first formula for the  $n$ th moment of  $X \sim PsL-R\{Exponential\}$  family follows from (2.9)

$$E(X^n) = \sum_{i=0}^{\infty} w_i^* E(Y_i^n) + \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} w_{i,k}^* E(Y_{i+k}^n) \tag{2.14}$$

Moments of some expo-G distributions are given by Nadarajah and Kotz [10] which can be used to obtain  $E(X^n)$

Secondly, the moment can be obtained from (2.10) above as

$$E(X^n) = \sum_{i=0}^{\infty} (i + 1) w_i^* I(n, i) + \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (i + k + 1) w_{i,k}^* I(n, i + k + 1) \tag{2.15}$$

where

$$I(n, i) = \int_0^1 Q^n(p) p^i dp$$

### 2.7 Moment generating function

The moment generating function  $M_X(t) = E(e^{tx})$  of a random variable  $X \sim PsL-R\{Exponential\}$  family of distributions can be obtained by (2.9)

$$M_X(t) = \sum_{i=0}^{\infty} w_i^* M_{Y_i}(t) + \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} w_{i,k}^* M_{Y_{i+k}}(t) \tag{2.16}$$

where  $M_{Y_i}(t)$  and  $M_{Y_{i+k}}(t)$  are the corresponding mgfs of  $Y_i$  and  $Y_{i+k}$

Secondly the mgf can be derived from (2.10)

$$E(X^n) = \sum_{i=0}^{\infty} (i + 1) w_i^* I_e(t, i) + \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (i + k + 2) w_{i,k}^* I_e(t, i + k + 1) \tag{2.17}$$

Where

$$I_e(t, i) = \int_0^1 e^{tQ(p)} p^i dp$$

### 3 Entropy

The Entropy of a random variable  $X$  with density function  $f(x)$  is the measure of variation of the uncertainty Renyi [11]. A large entropy value indicates greater uncertainty in the data. The Renyi entropy of random variable with density  $f(x)$  is defined as

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \left\{ \int_0^\infty f^\gamma(x) dx \right\} \tag{3.1}$$

for  $r > 0$ , and  $\gamma \neq 1$ . The Renyi entropy for  $X \sim PsL-R\{Exponential\}$  family is given by the following theorem.

Theorem 2: If a random variable  $X$  follows  $X \sim PsL-R\{Exponential\}$  family, then the Renyi entropy of  $X$   $I_R(\gamma)$  is given by

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \left[ \frac{\theta^{\gamma+k} (\beta-1)^{\gamma-k}}{\beta^\gamma} D_{j,k,i} I_{i,k,j} \right] \tag{3.2}$$

Proof.

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \int_0^\infty f^\gamma_X(x; \theta, \beta, \xi) dx$$

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \int_0^\infty \left[ \frac{\theta}{\beta} f_R(x; \xi) [(\beta-1) - \theta \log(1 - F_R(x; \xi))] [1 - F_R(x; \xi)]^{\theta-1} \right]^\gamma dx \tag{3.3}$$

From (4.2) we have

$$f_X^\gamma(x; \theta, \beta, \xi) = \left( \frac{\theta(\beta-1)}{\beta} \right)^\gamma f_R^\gamma(x; \xi) \left[ 1 - \frac{\theta}{(\beta-1)} \log(1 - F_R(x; \xi)) \right]^\gamma \times [1 - F_R(x; \xi)]^{\gamma(\theta-1)} \tag{3.4}$$

Using the power series in (2.6), (2.3) and (2.4) to simplify (4.3) we have

$$f_X^\gamma(x; \theta, \beta, \xi) = \left( \frac{\theta(\beta-1)}{\beta} \right)^\gamma \left( \frac{\theta}{\beta-1} \right)^k \sum_{j,k,i} (-1)^j \binom{\gamma(\theta-1)}{j} \binom{\gamma}{k} C_{k,i} \times F_R^{j+k+i}(x; \xi) f_R^\gamma(x; \xi) \tag{3.5}$$



$$\int_0^\infty f^\gamma_X(x; \theta, \beta, \xi) dx = \frac{\theta^{\gamma+k} (\beta-1)^{\gamma-k}}{\beta^\gamma} D_{j,k,i} \int_0^\infty F_R^{j+k+i}(x; \xi) f_R^\gamma(x; \xi) dx$$

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \left[ \frac{\theta^{\gamma+k} (\beta-1)^{\gamma-k}}{\beta^\gamma} D_{j,k,i} I_{i,k,j} \right]$$

Where

$$D_{j,k,i} = \sum_{j,k,i} (-1)^j \binom{\gamma(\theta-1)}{j} \binom{\gamma}{k} C_{k,i}$$

and

$$I_{i,k,j} = \int_0^\infty F_R^{j+k+i}(x; \xi) f_R^\gamma(x; \xi) dx$$

Shannon’s entropy for a random variable  $X$  with pdf  $f(x)$  is defined as  $E[-\log f(x)]$ . The Shannon entropy for  $PsL-R\{Exponential\}$  family is given by

$$E[-\log f_X(x; \theta, \beta, \xi)] = -\log \left( \frac{\theta(\beta-1)}{\beta} \right) - \log f_R(x; \xi) - \log \left[ 1 - \frac{\theta}{\beta-1} \log(1 - F_R(x; \xi)) \right] - (\theta-1) \log[1 - F_R(x; \xi)] \tag{3.6}$$

Following similar algebraic manipulations done in (3.4) we have (3.6) reduces to

$$E[-\log f_X(x; \theta, \beta, \xi)] = \log \beta - \log[\theta(\beta-1)] - E[f_R(x; \xi)] + \sum_{i,k} \frac{(-1)^{i+1}}{i+1} \left( \frac{\theta}{\beta-1} \right)^{i+1} \times C_{i+1,k} E[F_R^{i+k+1}(x; \xi)] + (\theta-1) \sum_{k=0}^\infty \frac{1}{k+1} E[F_R^{k+1}(x; \xi)] \tag{3.7}$$

where

$$C_{i+1,k} = (k+1)^{-1} \sum_{m=1}^k [m(i+2) - k] a_m C_{i+1,k-m}$$

### 4 Order Statistics

Order statistics has application in many areas of statistical theory and practice. Let  $X_1, X_2, \dots, X_n$  denote a random sample from the Pseudo –Lindley-R {exponential} family. The pdf  $f_{i:n}(x)$  of the  $i$ th order statistics  $X_{i:n}$  can be written as

$$\begin{aligned}
 f_{i:n}(x) &= \frac{n}{(i-1)!(n-i)!} f(x) F(x)^{i-1} [1-F(x)]^{n-i} \\
 &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x) F^{i+j-1}(x)
 \end{aligned}
 \tag{4.1}$$

Let  $f(x) = f_X(x)$  and  $F(x) = F_X(x)$  substituting for  $f_X(x)$  and  $F_X(x)$  in pdf of  $X_{i:n}$  for  $PsL-R\{Exponential\}$  distribution. Using (2.4) and (2.5), the  $i$ th order statistics for  $PsL-R\{Exponential\}$  family of distributions is given by

$$\begin{aligned}
 f_{i:n}(x) &= K_{i,j,z} \frac{\theta}{\beta^{z+1}} f_R(x; \xi) \left[ (\beta-1) \sum_{r=0}^{\infty} (-1)^r \binom{\theta-1}{r} F_R^r(x; \xi) \right. \\
 &\quad \left. + \theta \sum_{k,r} \frac{(-1)^r}{(K+1)} \binom{\theta-1}{r} F_R(x; \xi)^{k+r+1} \right] \\
 &\quad \times \left[ \left( \beta + \theta \sum_{w=0}^{\infty} \frac{F_R(x; \xi)^{w+1}}{w+1} \right)^z \left( \sum_{l=0}^{\infty} (-1)^l \binom{\theta}{l} F_R^l(x; \xi) \right)^z \right]
 \end{aligned}
 \tag{4.2}$$

Where

$$K_{i,j,z} = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \sum_{z=0}^{i+j-1} (-1)^z \binom{i+j-1}{z}$$

### 5 Maximum Likelihood Estimation

Given that  $x_1, x_2 \dots x_n$  are observed values from  $PsL-R\{Exponential\}$  family with parameters  $\theta, \beta$  and  $\xi$ . The likelihood function of  $\theta, \beta$  and  $\xi$  is given by

$$L(x, \theta, \beta, \xi) = \prod_{i=1}^n \left[ \frac{\theta}{\beta} f_R(x, \xi) [(\beta-1) - \theta \log(1 - F_R(x, \xi))] [1 - F_R(x, \xi)]^{\theta-1} \right]$$

Let  $L(x, \theta, \beta \text{ and } \xi) = l$ . The log likelihood function can be expressed as

$$\begin{aligned}
 l &= n[\log \theta - \log \beta] + \sum_{i=1}^n \log(f_R(x_i, \xi)) + \sum_{i=1}^n \log [(\beta-1) - \theta \log(1 - F_R(x_i, \xi))] \\
 &\quad + (\theta-1) \sum_{i=1}^n [\log [1 - F_R(x_i, \xi)]]
 \end{aligned}
 \tag{5.1}$$

Differentiating  $l$  partially with respect to  $\theta, \beta$  and  $\xi$  we have

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^n \left[ \frac{\log(1 - F_R(x_i; \xi))}{[(\beta-1) - \theta \log(1 - F_R(x_i; \xi))]} \right] + \sum_{i=1}^n [\log [1 - F_R(x_i; \xi)]]$$

$$\frac{\partial l}{\partial \beta} = \frac{-n}{\beta} + \sum_{i=1}^n \left[ \frac{1}{[(\beta-1) - \theta \log(1 - F_R(x_i; \xi))]} \right]$$

$$\frac{\partial l}{\partial \xi} = \sum_{i=1}^n \left[ \frac{1}{f_R(x_i; \xi)} \frac{\partial f_R(x_i; \xi)}{\partial \xi} \right] + \sum_{i=1}^n \left[ \frac{1}{[1 - F_R(x_i; \xi)]} \frac{\partial F_R(x; \xi)}{\partial \xi} \right]$$

### 6 Bivariate Extension

We introduce the bivariate extension of the proposed model in this section. A joint cdf of the proposed model is given by

$$F_{X,Y}(x, y) = 1 - \frac{(\beta - \theta \log(1 - F_R(x, y; \xi)))(1 - F_R(x, y; \xi))^\theta}{\beta} \tag{6.1}$$

$F_R(x, y; \xi)$  is the bivariate continuous distribution of the baseline distribution with marginal cdf's  $F_{R_1}(x; \xi)$  and  $F_{R_2}(y; \xi)$ . We call the (7.1) *Bivariate Pseudo-lindley-R{exponential}* distribution. The marginal cdfs are given by

$$F_X(x) = 1 - \frac{(\beta - \theta \log(1 - F_{R_1}(x; \xi)))(1 - F_{R_1}(x; \xi))^\theta}{\beta}$$

and

$$F_Y(y) = 1 - \frac{(\beta - \theta \log(1 - F_{R_2}(y; \xi)))(1 - F_{R_2}(y; \xi))^\theta}{\beta}$$

The joint pdf of  $(X, Y)$  is obtained from  $f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y}$

$$f_{X,Y}(x, y) = \frac{\theta}{\beta} A(x, y; \theta, \beta, \xi) [(\beta - 1) - \theta \log(1 - F_R(x, y; \xi))] [1 - F_R(x, y; \xi)]^{\theta-1}$$

Where

$$A(x, y; \theta, \beta, \xi) = \theta \frac{[(\beta - 1) - \theta \log(1 - F_R(x, y; \xi))]}{(1 - F_R(x, y; \xi))} \frac{\partial F_R(x, y; \xi)}{\partial x} \frac{\partial F_R(x, y; \xi)}{\partial y}$$

$$+ f_R(x, y; \xi) - \frac{\theta - 1}{(1 - F_R(x, y; \xi))} \frac{\partial F_R(x, y; \xi)}{\partial x} \frac{\partial F_R(x, y; \xi)}{\partial y}$$

The marginal pdfs are

$$f_x(x) = \frac{\theta}{\beta} f_{R_1}(x; \xi) [(\beta - 1) - \theta \log(1 - F_R(x; \xi))] [1 - F_R(x; \xi)]^{\theta - 1}$$

$$f_y(y) = \frac{\theta}{\beta} f_{R_2}(y; \xi) [(\beta - 1) - \theta \log(1 - F_{R_2}(y; \xi))] [1 - F_{R_2}(y; \xi)]^{\theta - 1}$$

The conditional cdf's are

$$F_{X/Y}(x/y) = \frac{1 - \left\{ (\beta - \theta \log(1 - F_R(x, y; \xi))) (1 - F_R(x, y; \xi))^\theta \right\} / \beta}{1 - \left\{ (\beta - \theta \log(1 - F_{R_2}(y; \xi))) (1 - F_{R_2}(y; \xi))^\theta \right\} / \beta}$$

$$F_{Y/X}(y/x) = \frac{1 - \left\{ (\beta - \theta \log(1 - F_R(x, y; \xi))) (1 - F_R(x, y; \xi))^\theta \right\} / \beta}{1 - \left\{ (\beta - \theta \log(1 - F_{R_1}(x; \xi))) (1 - F_{R_1}(x; \xi))^\theta \right\} / \beta}$$

The conditional density functions are

$$f_{X/Y}(x/y) = \frac{A(x, y; \theta, \beta, \xi) [(\beta - 1) - \theta \log(1 - F_R(x, y; \xi))] [1 - F_R(x, y; \xi)]^{\theta - 1}}{f_{R_2}(y; \xi) [(\beta - 1) - \theta \log(1 - F_{R_2}(y; \xi))] [1 - F_{R_2}(y; \xi)]^{\theta - 1}}$$

$$f_{Y/X}(y/x) = \frac{A(x, y; \theta, \beta, \xi) [(\beta - 1) - \theta \log(1 - F_R(x, y; \xi))] [1 - F_R(x, y; \xi)]^{\theta - 1}}{f_{R_1}(x; \xi) [(\beta - 1) - \theta \log(1 - F_{R_1}(x; \xi))] [1 - F_{R_1}(x; \xi)]^{\theta - 1}}$$

## 7 Special Pseudo – Lindley –R {Exponential} Distribution

By considering different R distributions we present some members of the proposed family of distributions.

### 7.1 The pseudo-lindley-weibull {exponential} distribution

The cdf and pdf of *Pseudo-Lindley-Weibull{Exponential}* distribution is obtained from (1.5) and (1.6)

respectively by taking the cdf of the baseline distribution  $F_R(x; \xi) = 1 - e^{-\left(\frac{x}{b}\right)^a}$ ;  $x, b, a > 0$

where  $\xi = (a, b)$ .

$$F_x(x; \theta, \beta, \xi) = 1 - \frac{\left( \beta + \theta \left( \frac{x}{b} \right)^a \right) \left( \exp \left( - \left( \theta \left( \frac{x}{b} \right)^a \right) \right) \right)}{\beta} \quad x, a, b, \theta > 0, \beta \geq 1$$

$$f_x(x; \theta, \beta, \xi) = \frac{a\theta}{b\beta} \left( \frac{x}{b} \right)^{a-1} \left[ (\beta - 1) + \theta \left( \frac{x}{b} \right)^a \right] e^{-\left(\frac{x}{b}\right)^a \theta} \quad x, a, b, \theta > 0, \beta \geq 1$$

For  $\beta = \theta + 1$ , *Pseudo-Lindley-Weibull{Exponential}* becomes Lindley Weibull distribution with parameters  $\theta, a$  and  $b$ . Cakmakyapan and Ozel [12]. The plots of the shapes of are *Pseudo-Lindley-Weibull{Exponential}* shown in Fig. 1 below. The plot in Fig. 1 shows that the density function of *Pseudo-Lindley-Weibull{Exponential}* distribution can be unimodal and increasing, while the hrf is also increasing.

### PsL-W{exponential} distribution

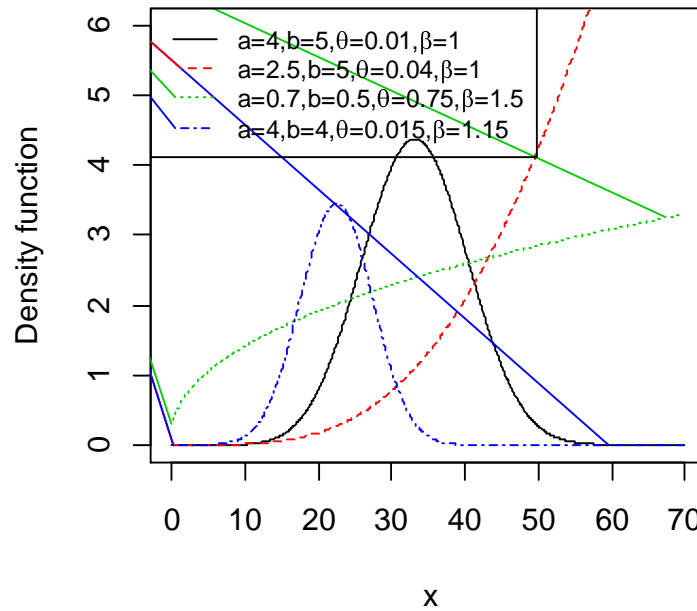


Fig. 1. Pdf plot of PsL-W{exponential} distribution for selected parameter values

### hrf of PsL-W{exponential} distribution

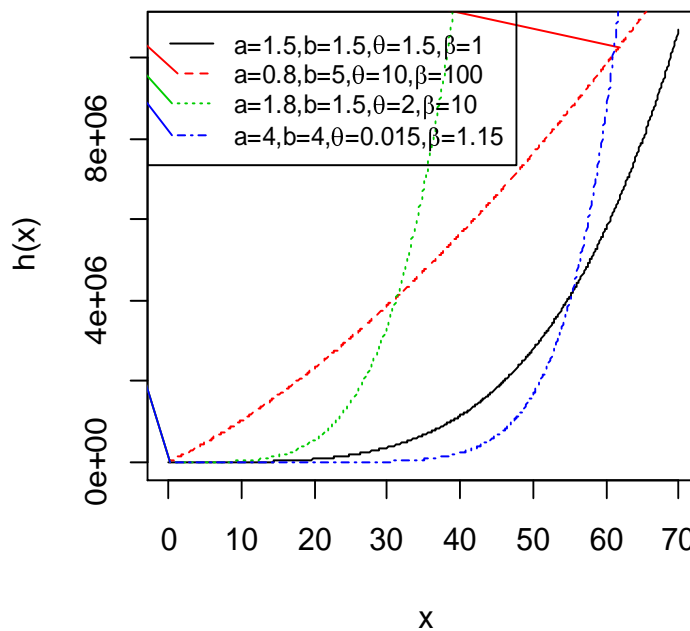


Fig. 2. hrf plot of PsL-W {exponential} distribution for selected parameter values

### 7.2 The pseudo-lindley-power {exponential} distribution

Letting the baseline distribution to be a power distribution with cdf  $F_R(x; \xi) = \left(\frac{x}{d}\right)^c; 0 \leq x \leq d$  and pdf

$f_R(x; \xi) = \frac{cx^{c-1}}{d^c}; 0 \leq x \leq d$ . The cdf and pdf of *Pseudo-Lindley-Power{Exponential}* distribution are given by

$$F_X(x; c, d, \xi) = 1 - \frac{\left(\beta - \theta \log\left(1 - \left(\frac{x}{d}\right)^c\right)\right) \left(1 - \left(\frac{x}{d}\right)^c\right)^\theta}{\beta}$$

and

$$f_X(x; c, d, \xi) = \frac{\theta cx^{c-1}}{\beta d^c} \left( (\beta - 1) - \theta \log\left(1 - \left(\frac{x}{d}\right)^c\right) \right) \left(1 - \left(\frac{x}{d}\right)^c\right)^{\theta-1}$$

The density function can be left-skewed, right-skewed, and symmetric and bathtub shaped, while the hrf is bathtub shaped and increasing.

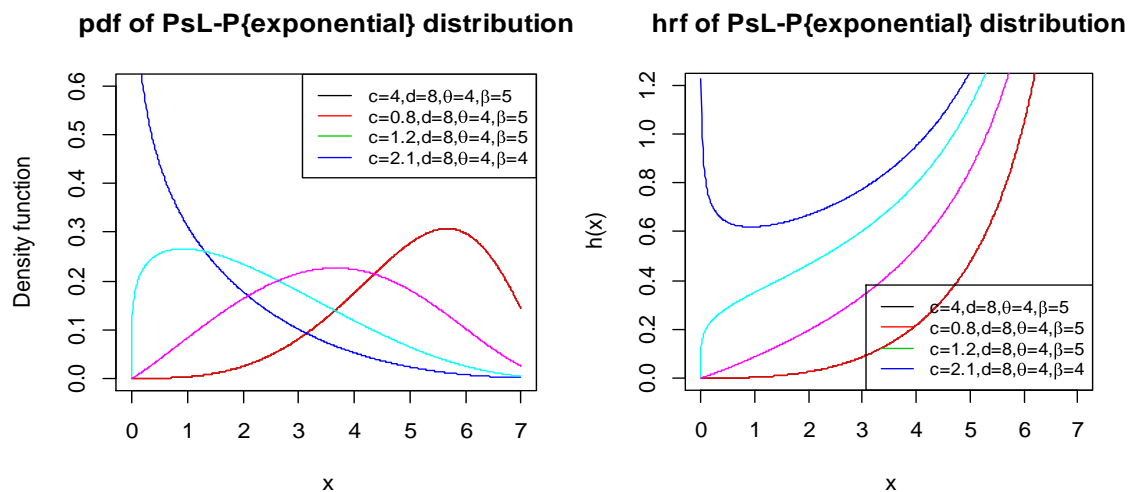


Fig. 3. Pdf and hrf plot of PsL-W {exponential} distribution for selected parameter values

## 8 Conclusion

There has been growing interest recently on generating families of distributions which are more flexible in modeling reliability and survival data. This paper adds to the growing literature in generalized family of distributions. We introduce a family of distributions called Pseudo-Lindley family of distribution based on the T-R{Y} framework proposed by Alzaatreh et al. [3]. The limiting behavior of the proposed family is examined and pdf expressed as exponentiated-G family. Some properties of the new family studied include quantile function, moments, entropy and order statistics. Maximum likelihood estimation method was used to derive the maximum likelihood estimates of parameters of the new family of distribution. Bivariate extension of Pseudo-Lindley family of distribution was also obtained. Plots of special members of the family showed that the density

could be right-skewed, left-skewed, symmetric and increasing, while the hazard rate function could be bathtub shaped and increasing.

## Competing Interests

Authors have declared that no competing interests exist.

## References

- [1] Zeghdoudi H, Nedjar S. A Pseudo Lindley Distribution and its Applications. Afrika Statistika. 2016;11(1):923-932.
- [2] Nedjar S, Zeghdoudi H. On Pseudo Lindley distribution: Properties and Applications. New Trends in Mathematical Sciences. 2017;5(1):59-65.
- [3] Alzaatreh A, Lee C, Famoye F. T – normal family of distributions: A new approach to generalize the normal distribution. J Statist Distributions and Applications. 2014;1:16.
- [4] Aljarrah MA, Lee C, Famoye F. A new method of generating T – X family of distributions using quantile functions. J Statist Distributions and Applications. 2014;1(2):17.
- [5] Gradshteyn IS, Ryzhik IM. Tables of Integrals, Series and Products. San Diego: Academic Press; 2000.
- [6] Mudholkar GS, Srivastava DK. Exponentiated Weibull family for analyzing bathtub failure-rate data. IEEE Transact. Reliab. 1993;42(2): 299 - 302.
- [7] Galton F. Enquires into human faculty and its development. Macmillan and company London; 1983.
- [8] Moor JJ. A quantile alternative for kurtosis. The Statistician. 1988;37:25-32.
- [9] Alizadeh M, Cordeiro G M, de Brito E, Demetrio CB. The beta Marshall-Olkin family of distributions. Journal of Statistical distributions and Applications. 2015; 2(4).
- [10] Nadaraja S, Kotz S. The exponentiated type distributions. Acta Applicandae Mathematicae. 2006;92,97 – 111.
- [11] Renyi A. On Measures of entropy and information. In Proceedings of the fourth Beckley Symposium on Mathematical Statistics and Probability. 1961;1(1):547-561.
- [12] Cakmakyapan S, Ozel G. The Lindley family of distributions: Properties and Applications: Hacettepe journal of Mathematics and Statistics. 2016;46(6):1114-1137.

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