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Generalized Weierstrass Integrability of a Class of Second-order Nonlinear Differential Equations

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The generalized Weierstrass integrability of a class of second-order nonlinear differential equations is considered. The conditions of existence and the corresponding expressions of generalized Weierstrass inverse integrating factors of the second-order nonlinear differential equation are presented. The relationship between the generalized Weierstrass inverse integrating factors and the Weierstrass inverse integrating factors is given. Finally, as an application of the main results, a Kudryashov-Sinelshchikov equation for obtaining traveling wave solutions is considered.

Keywords: Nonlinear differential equation; Generalized Weierstrass integrability; Weierstrass inverse integral factor.

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1 Introduction

Non-linear differential equations can be seen in a lot of research fields which include mathematics, physics, plasma physics, fluid mechanics, aerodynamics, atmosphere, ocean engineering, etc [1]. Research on the integrability of nonlinear differential equations not only helps us to understand the movement laws of various substances in physical phenomena under nonlinear action, but also plays an important role in the scientific explanation and engineering application of corresponding physical phenomena.

So far, there are many definitions of the integrability. We say that a certain nonlinear system is integrable, it must be specified in what sense. There are some definitions of integrability in the following senses. For a finite-dimensional dynamic system, there are Liouvillian integrability and Darboux integrability, strong integrability and weak integrability [2]. For nonlinear evolution systems, there are inverse scattering (IST) integrability, Lax integrability, symmetric integrability, Painlevé integrability [3] and so on. The connection between these integrability remains needing to be studied. In most cases, a system is integrable under one sense, it does not mean that it is also integrable under other senses. Therefore, research on every integrability of differential systems has certain theoretical and practical application significance.

In this paper, we focus on generalized Weierstrass integrability. The definitions of generalized Weierstrass integrability and Weierstrass integrability were introduced in [4, 5, 6, 7, 8, 9] and other papers. Now we restate them here. We say that a differential system is generalized Weierstrass integrable [4, 5, 6, 7, 8, 9] if it admits a first integral or an inverse integrating factor which is a generalized Weierstrass polynomial. We say that the differential system is Weierstrass integrable [4, 5, 6, 7, 8, 9] if it admits a first integral or an inverse integrating factor which is a Weierstrass polynomial. The generalized Weierstrass integrability of the Liénard differential system has been studied, see [5]. In [6], the Weierstrass and the generalized Weierstrass integrability of the Abel differential equation have been considered. Related research can also be found in other papers [7, 8, 9, 10, 11] and the references therein.

Generally speaking, we prefer to work with an inverse integrating factor than with a first integral, because it has better properties than first integral. So it is significant for us to obtain the inverse integrating factor of a system. In this paper we consider differential equations of the form

$$\begin{cases} \dot{x} = y = g(x,y) \\ \dot{y} = \sum\limits_{i=0}^{n} f_i(x)y^i = f(x,y) \end{cases} \tag{1.1}$$

where $f_i(x), i = 0, \cdots, n$, are meromorphic functions of x, and the dot denotes derivative with respect to the time t, real or complex. When $g(x,y) \equiv 1$ of (1.1), (1.1) becomes the system in [7]. The Weierstrass integrability of it has been studied in [7]. In [4], authors considered the Weierstrass integrability of system (1.1), and presented the expression and the condition of the existence of Weierstrass inverse integrating factor of system (1.1). In [4], the following inverse integrating factor form is considered,

$$u(x,y) = \sum_{i=0}^{s-1} u_i(x)y^i + y^s. \tag{1.2}$$

In this paper, we continue to consider the generalized Weierstrass integrability of system (1.1). We focus on the inverse integrating factor of following form

$$u(x,y) = \sum_{i=0}^{s} u_i(x)y^i,\ u_s(x) \not\equiv 0. \tag{1.3}$$

We obtain the condition and the corresponding expression of the generalized Weierstrass inverse integrating factor and give the relationship between the two different forms of inverse integrating factors. Finally, we make use of the results to obtain the traveling wave solutions or the equation satisfied by the traveling wave solutions of Kudryashov-Sinelshchikov equation.

2 Generalized Weierstrass inverse integrating factors

Associated to system (1.1), there is the differential operator

$$X = g\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}.$$

The vector field of system (1.1) is denoted as $P = (g, f)$.

For the integrity of this paper, we restate some definitions.

Definition 2.1. For system (1.1), If there is a non-constant continuous differentiable function $u(x, y)$ that satisfying $Xu = u \operatorname{div} P$, then $u(x, y)$ is an inverse integrating factor of system (1.1), where div is the divergence operation.

For system (1.1), If there is a non-constant continuous differentiable function $\mu(x, y)$ that satisfying $\operatorname{div}(\mu P) = 0$, then $\mu(x, y)$ is an integrating factor of system (1.1). It is easy to know that if $u(x, y) \neq 0$ is an inverse integrating factor of system (1.1), $1/u(x, y)$ has to be an integrating factor of system(1.1).

Definition 2.2. [4-9]Let $\mathbb{C}[[x]]$ be the set of the formal power series in the variable x with coefficients in $\mathbb{C}$, and $\mathbb{C}[y]$ the set of the polynomials in the variable y with coefficients in $\mathbb{C}$. A polynomial of the form

$$\sum_{i=0}^{n} f_i(x)y^i \in \mathbb{C}[[x]]C[y], \tag{2.1}$$

is called a formal Weierstrass polynomial in y of degree n if and only if $f_n(x) = 1$ and $f_i(0) = 0$ for $i < n$. A formal polynomial whose coefficients are convergent is called Weierstrass polynomial. A polynomial of the form (2.1) is called a formal generalized Weierstrass polynomial in y of degree n if and only if $f_n(x) \not\equiv 0$. A formal polynomial whose coefficients are convergent is called generalized Weierstrass polynomial.

Then our main results are listed as follows. The results of system (1.1) with $n = s = 2$ having Weierstrass inverse integrating factors have been given in theorem 1.2 in [4].

Theorem 2.1. *For system* (1.1) *with* $n = 2$,

(1) *When* $s = 1$ *and* $f_0(x) = cf_1(x)e^{\int f_2(x)dx}$, *where* c *is an arbitrary constant, then*

$$u(x, y) = e^{\int f_2(x)dx}y + ce^{2\int f_2(x)dx}.$$

(2) *When* $s = 2$, *system* (1.1) *has a Weierstrass inverse integrating factor if and only if either* $f_0(x) = f_1(x) = 0$, *and then* $u(x, y) = y^2$,
or $f_0(x) = 0$ *and* $f_1(x) \neq 0$, *and then*

$$u(x, y) = y^2 + (-e^{\int f_2(x)dx}\int f_1(x)e^{-\int f_2(x)ds}dx)y,$$

or $f_1(x) = 0$ *and* $f_0(x) \neq 0$, *and then*

$$u(x, y) = y^2 - 2e^{2\int f_2(x)dx}\int f_0(x)e^{-2\int f_2(x)ds}dx,$$

or $f_0(x)f_1(x) \neq 0$ *with* $f_0(x)f_1(x)f_2(x) + f_1'(x)f_0(x) - f_0'(x)f_1(x) \neq 0$ *and*

$$\begin{aligned} &f_0'(x)f_1(x)^2 f_2(x) + f_0(x)f_1(x)^2 f_2'(x) - f_0''(x)f_1(x)^2 - 3f_0(x)f_1'(x)^2 \\ &+3f_0'(x)f_1'(x)f_1(x) + f_1''(x)f_0(x)f_1(x) - 2f_1'(x)f_0(x)f_1(x)f_2(x) = 0, \end{aligned}$$

then $u(x,y) = u_0(x) + u_1(x)y + y^2$, *where* $u_0(x)$ *and* $u_1(x)$ *are listed as follows,*

$$u_0(x) = \frac{f_1(x)f_0(x)^2}{f_0(x)f_1(x)f_2(x) + f_0(x)f_1'(x) - f_0'(x)f_1(x)},$$

$$u_1(x) = \frac{f_0(x)f_1(x)^2}{f_0(x)f_1(x)f_2(x) + f_0(x)f_1'(x) - f_0'(x)f_1(x)}.$$

Proof. For system (1.1) with $n = 2$, when system (1.1) has the Weierstrass inverse integrating factor of the form (1.3), based on the definition of inverse integrating factor, we can get the following formula,

$$\begin{aligned} &\sum_{i=0}^{s} u_i'(x)y^{i+1} + \sum_{i=0}^{s} iu_i(x)y^{i-1}\left(f_2(x)y^2 + f_1(x)y + f_0(x)\right) \\ &= (2f_2(x)y + f_1(x))\,(\sum_{i=0}^{s} u_i(x)y^i). \end{aligned} \tag{2.2}$$

Now, combining the coefficients of y^{s+1}, y^s, $y^{s-i}(i = 2, \cdots, s-1)$, y^0 in (2.2), and letting them be zeroes, we get

$$\begin{aligned} &u_s'(x) + (s-2)u_s(x)f_2(x) = 0, \\ &u_{s-1}'(x) + (s-1)u_s(x)f_1(x) + (s-3)u_{s-1}(x)f_2(x) = 0, \\ &u_{s-2}'(x) + su_s(x)f_0(x) + (s-2) \\ &u_{s-1}(x)f_1(x) + (s-4)u_{s-2}(x)f_2(x) = 0, \\ &u_{s-i-1}'(x) + (s-i+1)u_{s-i+1}(x)f_0(x) + (s-i-1) \\ &u_{s-i}(x)f_1(x) + (s-i-3)u_{s-i-1}(x)f_2(x) = 0, \\ &u_1(x)f_0(x) - u_0(x)f_1(x) = 0. \end{aligned} \tag{2.3}$$

(1) When $s = 1$, by (2.3), we have

$$\begin{aligned} &u_1'(x) - u_1(x)f_2(x) = 0, \\ &u_0'(x) - 2u_0(x)f_2(x) = 0, \\ &u_1(x)f_0(x) - u_0(x)f_1(x) = 0. \end{aligned} \tag{2.4}$$

Solving (2.4), we obtain

$$\begin{aligned} &u_1(x) = e^{\int f_2(x)dx}, \\ &u_0(x) = ce^{2\int f_2(x)dx}, \\ &f_0(x) = cf_1(x)e^{\int f_2(x)dx}, \end{aligned}$$

where c is an arbitrary constant. Therefore, the Weierstrass inverse integrating factor of system (1.1) is

$$u(x,y) = e^{\int f_2(x)dx}y + e^{2\int f_2(x)dx}$$

under the condition

$$f_0(x) = cf_1(x)e^{\int f_2(x)dx}.$$

(2) When $s = 2$, by the first equation of (2.3), we have

$$u_2'(x) = 0.$$

So we can suppose $u_2(x) \equiv 1$, The corresponding inverse integrating factor is the form (1.2). The relevant results have been given and proved in [4], we will not prove it here. □

Owing to space limitation, we will provide system (1.1) with $n = 1$ and $n = 3$ admitting generalized Weierstrass inverse integrating factor in another paper. Next we consider system (1.1) with $n > 3$ has a generalized Weierstrass inverse integrating factor. First, we introduce two lemmas as follows.

Lemma 2.2. *Equation* (2.5) *can be written as* (2.6)

$$\begin{aligned}&\sum_{i=0}^{n} u_i'(x)y^{i+1} + \sum_{i=0}^{n} iu_i(x)y^{i-1}(f_n(x)y^n + \cdots + f_1(x)y + f_0(x))\\&= (nf_n(x)y^{n-1} + \cdots + 2f_2(x)y + f_1(x))(\sum_{i=0}^{n} u_i(x)y^i),\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\sum_{i=0}^{n} u_i'(x)y^{i+1} + f_0(x)\sum_{i=0}^{n-1}(i+1)u_{i+1}(x)y^i\\&+\sum_{i=0}^{2n-2} y^l \sum_{i=\max\{0,l+1-n\}}^{\min\{l,n\}} (2i-1-l)u_i(x)f_{l-i+1}(x) = 0.\end{aligned} \tag{2.6}$$

Relevant proof has been given in [4], we will not prove it here.

Lemma 2.3.

$$S_{m,j}(x) = \sum_{j=1}^{l-1}(l-2j)f_{m-j}(x)f_{m-l+j}(x) = 0.(m, l \le n+2)$$

Relevant proof has been given in [4], we will not prove it here.
Based on Lemma 2.3, we can get easily

$$S_{n+1,n+1}(x) = \sum_{j=1}^{n}(n+1-2j)f_{n+1-j}(x)f_j(x) = 0 \tag{2.7}$$

and

$$S_{n+2,n+2}(x) = \sum_{j=1}^{n+1}(n+2-2j)f_{n+2-j}(x)f_j(x) = 0. \tag{2.8}$$

Theorem 2.4. *System* (1.1) *with* $n > 3$ *has a Weierstrass inverse integrating factor of the form* (1.3)*, where*

$$u_{n-k+1}(x) = \frac{f_{n-k+1}(x)}{f_n(x)}u_n(x), (k = 2, \cdots, n-2) \tag{2.9}$$

$$u_0(x) = \frac{A(x)}{n(n-2)(n-1)f_n(x)^4}, \tag{2.10}$$

$$u_1(x) = \frac{B(x)}{(n-2)(n-1)f_n(x)^3}, \tag{2.11}$$

$$u_2(x) = \frac{(n-2)f_2(x)u_n(x) + u_n'(x)}{(n-2)f_n(x)}, \tag{2.12}$$

$$u_n(x) = ce^{\int \frac{C(x)}{D(x)}dx}, \tag{2.13}$$

where

$$
\begin{aligned}
A(x) = &\left((n-2)(n-1)f_n(x)^2\left(f'_{n-2}(x)+nf_n(x)f_0(x)\right)\right.\\
&\left.-(n-2)^2f_{n-1}(x)\left(f'_{n-1}(x)f_n(x)-f_{n-1}(x)f'_n(x)\right)\right)u_n(x)\\
&+\left(2(n-1)f_n(x)^2f_{n-2}(x)-(n-2)f_{n-1}(x)^2f_n(x)\right)u'_n(x),\\
B(x) = &\left((n-2)\left(f_n(x)f'_{n-1}(x)-f'_n(x)f_{n-1}(x)\right.\right.\\
&\left.\left.+(n-1)f_n(x)^2f_1(x)\right)\right)u_n(x)+f_n(x)f_{n-1}(x)u'_n(x),\\
C(x) = &(n-2)\left((n-2)f_1(x)f_{n-1}(x)+nf_0(x)f_n(x)\right)\left(f'_n(x)f_{n-1}(x)\right.\\
&\left.-\ f_n(x)f'_{n-1}(x)\right),\\
D(x) = &f_n(x)^2\left(n(n-3)f_0(x)f_{n-1}(x)+2(n-1)f_1(x)f_{n-2}(x)\right)\\
&-(n-2)f_n(x)f_{n-1}(x)\left(nf_0(x)f_n(x)+f_1(x)f_{n-1}(x)\right).
\end{aligned}
$$

Proof. For system (1.1), when it has a Weierstrass inverse integrating factor of (1.3), based on the definition of inverse integrating factor, we can get the following formula,

$$
\begin{aligned}
&\sum_{i=0}^{s}u'_i(x)y^{i+1}+\sum_{i=0}^{s}iu_i(x)y^{i-1}(f_n(x)y^n+\cdots+f_1(x)y+f_0(x))\\
&=(nf_n(x)y^{n-1}+\cdots+2f_2(x)y+f_1(x))(\sum_{i=0}^{s}u_i(x)y^i).
\end{aligned}
\tag{2.14}
$$

Computing the coefficients of y^{s+n-1} in (2.14) and letting them be zeroes, we get

$$
su_s(x)f_n(x)=nu_s(x)f_n(x),
$$

So $s=n$. Using Lemma 2.2, (2.14) can be written as

$$
\begin{aligned}
&\sum_{i=0}^{n}u'_i(x)y^{i+1}+f_0(x)\sum_{i=0}^{n-1}(i+1)u_{i+1}(x)y^i\\
&+\sum_{i=0}^{2n-2}y^l\sum_{i=\max\{0,l+1-n\}}^{\min\{l,n\}}(2i-1-l)u_i(x)f_{l-i+1}(x)=0.
\end{aligned}
\tag{2.15}
$$

Now we prove (2.9). Computing the coefficients of y^l,$(l=n+2,\cdots,2n-2)$ in (2.15), we have

$$
\sum_{i=l+1-n}^{n}(2i-1-l)u_i(x)f_{l-i+1}(x)=0. \tag{2.16}
$$

Letting $l=2n-k$, $(k=2,\cdots,n-2)$, (2.16) becomes

$$
\sum_{i=n+1-k}^{n}(2i-2n+k-l)u_i(x)f_{2n-k+1-i}(x)=0. \tag{2.17}
$$

Similarly, writing $i=n-j$, (2.17) can be

$$
\sum_{j=0}^{k-1}(k-2j-1)u_{n-j}(x)f_{n-k+j+1}(x)=0,
$$

that is,

$$\begin{aligned}&(k-1)(u_n(x)f_{n-k+1}(x)-u_{n-k+1}(x)f_n(x))\\&+\sum_{j=1}^{k-2}(k-2j-1)u_{n-j}(x)f_{n-k+j+1}(x)=0.\end{aligned}\tag{2.18}$$

Next, we prove (2.9) by mathematical induction method.
When $k=2$, using (2.18), we can have

$$u_n(x)f_{n-1}(x)-u_{n-1}(x)f_n(x)=0,$$

so

$$u_{n-1}(x)=\frac{f_{n-1}(x)u_n(x)}{f_n(x)}.\tag{2.19}$$

We assume (2.9) is true for $k=3,\cdots,n-3$. So, one has the formula

$$u_{n-k+1}(x)=\frac{f_{n-k+1}(x)}{f_n(x)}u_n(x).(k=3,\cdots,n-3)\tag{2.20}$$

For $k=n-2$, using (2.18), we get

$$(n-3)(u_n(x)f_3(x)-u_3(x)f_n(x))+\sum_{j=1}^{n-4}(n-2j-3)u_{n-j}(x)f_{3+j}(x)=0.\tag{2.21}$$

Substituting (2.19) and (2.20) into (2.21), we get

$$\begin{aligned}&(n-3)(u_n(x)f_3(x)-u_3(x)f_n(x))\\&+\frac{u_n(x)}{f_n(x)}\sum_{j=1}^{n-4}(n-2j-3)f_{n-j}(x)f_{3+j}(x)=0.\end{aligned}\tag{2.22}$$

By Lemma 2.3 , (2.22) becomes

$$(n-3)(u_n(x)f_3(x)-u_3(x)f_n(x))=0,$$

that is

$$u_3(x)=\frac{f_3(x)}{f_n(x)}u_n(x).$$

This concludes the proof of (2.9).

Computing the coefficient of y^{n+1},y^n,y^{n-1} in (2.15), we get

$$u_n'(x)+\sum_{i=2}^{n}(2i-2-n)u_i(x)f_{n-i+2}(x)=0,\tag{2.23}$$

$$u_{n-1}'(x)+\sum_{i=1}^{n}(2i-1-n)u_i(x)f_{n-i+1}(x)=0,\tag{2.24}$$

$$u_{n-2}'(x)+nu_n(x)f_0(x)+\sum_{i=0}^{n-1}(2i-n)u_i(x)f_{n-i}(x)=0.\tag{2.25}$$

Now, with the aids of (2.8) and (2.9), (2.23) becomes

$$
\begin{aligned}
&u_n'(x) + (2-n)f_n(x)u_2(x) + \sum_{i=3}^{n-1}(2i-n-2)u_i(x)f_{n-i+2}(x) \\
&+(n-2)u_n(x)f_2(x) \\
=&u_n'(x) + (2-n)u_2(x)f_n(x) - \frac{u_n(x)}{f_n(x)}\sum_{i=3}^{n-1}(n+2-2i)f_i(x)f_{n-i+2}(x) \\
&+(n-2)u_n(x)f_2(x) \\
=&u_n'(x) - \frac{u_n(x)}{f_n(x)}(\sum_{i=1}^{n+1}(n+2-2i)f_i(x)f_{n+2-i}(x) - nf_1(x)f_{n+1}(x) \\
&-(n-2)f_2(x)f_n(x) - (2-n)f_n(x)f_2(x) + nf_{n+1}(x)f_1(x)) \\
&+(2-n)u_2(x)f_n(x) + (n-2)u_n(x)f_2(x) \\
=&u_n'(x) + (2-n)u_2(x)f_n(x) + (n-2)u_n(x)f_2(x) - \frac{u_n(x)}{f_n(x)}S_{n+2,n+2}(x) \\
=&u_n'(x) + (2-n)u_2(x)f_n(x) + (n-2)u_n(x)f_2(x) \\
=&0.
\end{aligned}
$$

Therefore,

$$u_2(x) = \frac{(n-2)f_2(x)u_n(x) + u_n'(x)}{(n-2)f_n(x)},$$

that is (2.12).

Similarly, with the aids of (2.7) and (2.9), (2.24) can be rewritten as

$$
\begin{aligned}
&u_{n-1}'(x) + (1-n)u_1(x)f_n(x) + (n-1)u_n(x)f_1(x) + (3-n)u_2(x)f_{n-1}(x) \\
&-\sum_{i=3}^{n-1}(n+1-2i)u_i(x)f_{n-i+1}(x) \\
=&u_{n-1}'(x) + (1-n)u_1(x)f_n(x) + (n-1)u_n(x)f_1(x) + (3-n)u_2(x)f_{n-1}(x) \\
&-\frac{u_n(x)}{f_n(x)}\sum_{i=3}^{n-1}(n+1-2i)f_i(x)f_{n-i+1}(x) \\
=&u_{n-1}'(x) + (1-n)u_1(x)f_n(x) + (n-1)u_n(x)f_1(x) + (3-n)u_2(x)f_{n-1}(x) \\
&-\frac{u_n(x)}{f_n(x)}(\sum_{i=1}^{n}(n+1-2i)f_i(x)f_{n+1-i}(x) - (1-n)f_1(x)f_n(x) \\
&-(n-1)f_1(x)f_n(x) - (n-3)f_{n-1}(x)f_2(x)) \\
=&u_{n-1}'(x) + (1-n)u_1(x)f_n(x) + (n-1)u_n(x)f_1(x) + (3-n)u_2(x)f_{n-1}(x) \\
&-\frac{u_n(x)}{f_n(x)}(S_{n+1,n+1}(x) - (n-3)f_{n-1}(x)f_2(x)) \\
=&u_{n-1}'(x) + (1-n)u_1(x)f_n(x) + (n-1)u_n(x)f_1(x) + (3-n)u_2(x)f_{n-1}(x) \\
&-\frac{u_n(x)}{f_n(x)}(3-n)f_{n-1}(x)f_2(x) \\
=&0.
\end{aligned}
\tag{2.26}
$$

Substituting (2.12) and (2.19) into (2.26), we have

$$u_1(x) = \frac{B(x)}{(n-2)(n-1)f_n(x)^3},$$

where

$$B(x) = \left((n-2)\left(f_n(x)f'_{n-1}(x) - f'_n(x)f_{n-1}(x)\right.\right.$$
$$\left.\left.+(n-1)f_n(x)^2 f_1(x)\right)\right)u_n(x) + f_n(x)f_{n-1}(x)u'_n(x),$$

that is (2.11).

Similarly, (2.25) can be rewritten as

$$\begin{aligned}
&u'_{n-2}(x) + nf_0(x)u_n(x) + \sum_{i=3}^{n-1}(2i-n)u_i(x)f_{n-i}(x) + (2-n)u_1(x)f_{n-1}(x)\\
&+(4-n)u_2(x)f_{n-2}(x) - nu_0(x)f_n(x)\\
&=u'_{n-2}(x) - \frac{u_n(x)}{f_n(x)}\sum_{i=3}^{n-1}(n-2i)f_i(x)f_{n-i}(x) + (2-n)u_1(x)f_{n-1}(x)\\
&+nf_0(x)u_n(x) - nu_0(x)f_n(x) + (4-n)u_2(x)f_{n-2}(x)\\
&=u'_{n-2}(x) - \frac{u_n(x)}{f_n(x)}(\sum_{i=1}^{n-1}(n-2i)f_i(x)f_{n-i}(x) - (n-2)f_1(x)f_{n-1}(x)\\
&-(n-4)f_2(x)f_{n-2}(x)) + nf_0(x)u_n(x) - nf_n(x)u_0(x)\\
&+(2-n)u_1(x)f_{n-1}(x) + (4-n)u_2(x)f_{n-2}(x)\\
&=u'_{n-2}(x) - \frac{u_n(x)}{f_n(x)}(S_{n,n}(x) - (n-2)f_1(x)f_{n-1}(x) - (n-4)f_2(x)f_{n-2}(x))\\
&+nf_0(x)u_n(x) - nf_n(x)u_0(x) + (2-n)u_1(x)f_{n-1}(x) + (4-n)u_2(x)f_{n-2}(x)\\
&=u'_{n-2}(x) - \frac{u_n(x)}{f_n(x)}\left((2-n)f_1(x)f_{n-1}(x) - (n-4)f_2(x)f_{n-2}(x)\right)\\
&+nf_0(x)u_n(x) - nf_n(x)u_0(x) + (2-n)u_1(x)f_{n-1}(x)\\
&+(4-n)u_2(x)f_{n-2}(x)\\
&=0.
\end{aligned} \tag{2.27}$$

Substituting (2.12), (2.11) and (2.9) with $k = 3$ into (2.27), we have

$$u_0(x) = \frac{A(x)}{n(n-2)(n-1)f_n(x)^4},$$

where

$$\begin{aligned}
A(x) = &\left((n-2)(n-1)f_n(x)^2(f'_{n-2}(x) + nf_n(x)f_0(x))\right.\\
&\left.-(n-2)^2 f_{n-1}(x)(f'_{n-1}(x)f_n(x) - f_{n-1}(x)f'_n(x))\right)u_n(x)\\
&+\left(2(n-1)f_n(x)^2 f_{n-2}(x) - (n-2)f_{n-1}(x)^2 f_n(x)\right)u'_n(x),
\end{aligned}$$

that is (2.10).

Now computing the coefficients of y^n in (2.15), we have

$$u_1(x)f_0(x) = u_0(x)f_1(x). \tag{2.28}$$

Substituting (2.11), (2.10) into (2.28), we have

$$u'_n(x) - \frac{C(x)}{D(x)}u_n(x) = 0.$$

It is easy to get

$$u_n(x) = ce^{\int \frac{C(x)}{D(x)}dx},$$

where c is an arbitrary constant and

$$\begin{aligned} C(x) &= (n-2)\left((n-2)f_1(x)f_{n-1}(x) + nf_0(x)f_n(x)\right)\left(f_n'(x)f_{n-1}(x)\right.\\ &\quad \left.-\ f_n(x)f_{n-1}'(x)\right),\\ D(x) &= f_n(x)^2\left(n(n-3)f_0(x)f_{n-1}(x) + 2(n-1)f_1(x)f_{n-2}(x)\right)\\ &\quad -(n-2)f_n(x)f_{n-1}(x)\left(nf_0(x)f_n(x) + f_1(x)f_{n-1}(x)\right), \end{aligned}$$

that is (2.13). □

Theorem 2.5. *The differential equations of the form*

$$\begin{cases} \dot{x} = p(x)y + q(x), \\ \dot{y} = \sum\limits_{i=0}^{n} f_i(x)y^i, \end{cases} \tag{2.29}$$

where $p(x) \neq 0$ and $f_i(x), i = 0, \cdots, n$, are meromorphic functions of x, and the dot denotes derivative with respect to the time t, real or complex, can be transformed into the system (1.1) *doing the change of variables $y = \frac{1}{p(x)}u - \frac{q(x)}{p(x)}$.*

Proof. Let $y = \frac{1}{p(x)}u - \frac{q(x)}{p(x)}$, we have

$$\dot{x} = p(x)(\frac{1}{p(x)}u - \frac{q(x)}{p(x)}) + q(x) = u.$$

Because $u = (y + \frac{q(x)}{p(x)})p(x) = p(x)y + q(x)$, we have

$$\begin{aligned} &\dot{u} = p'(x)y + p(x)y' + q'(x)\\ &= p'(x)(\tfrac{1}{p(x)}u - \tfrac{q(x)}{p(x)}) + q'(x) + p(x)\left(f_n(x)y^n + \cdots + f_1(x)y + f_0(x)\right)\\ &= \tfrac{p'(x)}{p(x)}u - \tfrac{p'(x)q(x)}{p(x)} + q'(x) + p(x)(f_n(x)(\tfrac{1}{p(x)}u - \tfrac{q(x)}{p(x)})^n + \cdots\\ &+f_1(x)(\tfrac{1}{p(x)}u - \tfrac{q(x)}{p(x)}) + f_0(x))\\ &= b_n(x)u^n + \cdots + b_1(x)u + b_0(x). \end{aligned}$$

Therefore, (2.29) can be rewritten as

$$\begin{cases} \dot{x} = u, \\ \dot{u} = \sum\limits_{i=0}^{n} b_i(x)u^i. \end{cases}$$

It is the form of system (1.1), thus it will have a series of related results of Theorem 2.4. □

If system (1.1) is generalized Weierstrass integrable, it means that system (1.1) has the inverse integrating factor of the form

$$u_1(x, y) = \sum_{i=0}^{s} u_i(x)y^i, u_s(x) \not\equiv 0.$$

Let

$$u_1(x, y) = u_s(x)\sum_{i=0}^{s}\frac{u_i(x)}{u_s(x)}y^i = u_s(x)u_2(x, y), \tag{2.30}$$

where $u_2(x, y)$ is the form (1.2). We have the following result.

Theorem 2.6. *Suppose $u_1(x, y)$ is an inverse integrating factor, $u_2(x, y)$ is an inverse integrating factor of system (1.1) if and only if $u_s(x)$ is a first integral of system (1.1).*

Proof. $u_1(x, y)$ is an inverse integrating factor of system (1.1), so

$$Xu_1(x, y) = u_1(x, y) \operatorname{div} P. \tag{2.31}$$

Plugging (2.30) in (2.31), we have

$$u_s(x)Xu_2(x, y) + u_2(x, y)Xu_s(x) = u_s(x)u_2(x, y) \operatorname{div} P. \tag{2.32}$$

If $u_2(x, y)$ is an inverse integrating factor of system (1.1). then

$$Xu_2(x, y) = u_2(x, y) \operatorname{div} P.$$

Substituting the above formula into (2.32), we have

$$u_2(x, y)Xu_s(x) = 0.$$

Therefore, $u_s(x)$ is a first integral of system (1.1).

If $u_s(x)$ is a first integral of system (1.1), we have

$$Xu_s(x) = 0. \tag{2.33}$$

By (2.30), we have

$$Xu_2(x, y) = X\frac{u_1(x, y)}{u_s(x)} = \frac{u_s(x)Xu_1(x, y) - u_1(x, y)Xu_s(x)}{u_s(x)^2}. \tag{2.34}$$

Substituting (2.33) into (2.34), we have

$$Xu_2(x, y) = \frac{Xu_1(x, y)}{u_s(x)} = \frac{u_1(x, y) \operatorname{div} P}{u_s(x)} = u_2(x, y) \operatorname{div} P.$$

Therefore $u_2(x, y)$ is an inverse integrating factor of system (1.1), The proof is finished. □

3 Application on a Kudryashov-Sinelshchikov equation

We consider the following Kudryashov-Sinelshchikov equation

$$u_t + auu_x + bu_{xxx} + k\left(uu_{xx}\right)_x + mu_xu_{xx} + nu_{xx} + f\left(u_x^2 + uu_{xx}\right) = 0,$$

where u denotes the density and a, b, k, m, n and f are real parameters. It is a nonlinear partial differential equation that describes the pressure waves in a mixture liquid with gas. Research on the spread of the pressure waves is one of the most important problems in physics. Finding exact solutions of the nonlinear differential equations plays an essential role in the study of the corresponding nonlinear physical phenom- ena because it can help us understand the physical phenomena. Up to now, researchers have succeeded in applying several methods to study the Kudryashov-Sinelshchikov equation and getting some results. Some traveling wave solutions, periodic solutions, group invariant solutions, analytical power series solutions and other exact solutions have been obtained [12, 13, 14, 15].

Here, we consider the equation with coefficient $k = 0$ and $bm \neq 0$ that is

$$u_t + auu_x + bu_{xxx} + mu_xu_{xx} + nu_{xx} + f\left(u_x^2 + uu_{xx}\right) = 0. \tag{3.1}$$

We suppose the wave transformations

$$u(x,t) = u(\xi), \quad \xi = x - ct, \tag{3.2}$$

where $c \in R$ is the wave speed. Plugging (3.2) into (3.1) and integrating the equation once, (3.1) can be converted into the following form,

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -\frac{m}{2b}v^2 - \left(\frac{f}{b}u + \frac{n}{b}\right)v + \left(\frac{c}{b}u - \frac{a}{2b}u^2\right). \end{cases} \tag{3.3}$$

(3.3) is the form of system (1.1) with $n = 2$. where

$$f_2(u) = -\frac{m}{2b}, \quad f_1(u) = -\frac{f}{b}u - \frac{n}{b}, \quad f_0(u) = \frac{c}{b}u - \frac{a}{2b}u^2.$$

Based on Theorem 2.1, we can have the following results.

(1) When $s = 1$ and $f_0(u) = f_1(u) = 0$, we have a Weierstrass inverse integrating factor

$$\mu(u,v) = e^{-\frac{m}{2b}u}v + e^{-\frac{m}{b}u}. \tag{3.4}$$

In fact, (3.3) becomes

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -\frac{m}{2b}v^2. \end{cases} \tag{3.5}$$

According to the Weierstrass inverse integration factor (3.4), it is easy to know the solution of (3.5) is

$$-e^{\frac{m}{2b}u}v + \ln\left|v + e^{-\frac{m}{2b}u}\right| + \frac{m}{2b}u = c_0,$$

where $u' = v$ and c_0 is an arbitrary constant. Then, we get an equation satisfied by traveling wave solutions

$$-e^{\frac{m}{2b}u}u' + \ln\left|u' + e^{-\frac{m}{2b}u}\right| + \frac{m}{2b}u = c_0.$$

(2) When $s = 2$ and $f_0(u) = f_1(u) = 0$, we have a Weierstrass inverse integrating factor

$$\mu(u,v) = v^2. \tag{3.6}$$

In fact, (3.3) becomes

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -\frac{m}{2b}v^2. \end{cases} \tag{3.7}$$

According to the Weierstrass inverse integration factor (3.6), it is easy to know the solution of (3.7) is

$$-\frac{m}{2b}u - \ln v = c_1,$$

where $u' = v$ and c_1 is an arbitrary constant. Solving it, we can obtain the exact solution of the corresponding (3.1)

$$u = \frac{2b}{m}\ln|\xi| + c_2 = \frac{2b}{m}\ln|x| + c_2,$$

where c_2 is an arbitrary constant.

(3) When $s = 2$ and $f_0(u) = 0$, we have a Weierstrass inverse integrating factor

$$\mu(u,v) = \frac{2fmu - 4bf + 2nm}{m^2}v + v^2. \tag{3.8}$$

In fact, (3.3) becomes

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -\frac{m}{2b}v^2 - \left(\frac{f}{b}u + \frac{n}{b}\right)v. \end{cases} \tag{3.9}$$

According to the Weierstrass inverse integration factor (3.8), it is easy to know the solution of (3.9) is

$$-\frac{m}{2b}u+\ln\left|\frac{1}{16b^2f^2m^2u+8b^2m^3fv-32b^3f^2m+16b^2fnm^2}\right|=c_3,$$

where $u'=v$ and c_3 is an arbitrary constant. Solving it, we can obtain the exact solution of the corresponding (3.1),

$$u=c_4e^{-\frac{2f}{m}\xi}+\frac{2bf-mn}{fm}=c_4e^{-\frac{2fx}{m}}+\frac{2bf-mn}{fm},$$

where c_4 is an arbitrary constant.

(4) When $s=2$ and $f_1(u)=0$, we have a Weierstrass inverse integrating factor

$$\mu(u,v)=\frac{am^2u^2-(2m^2c+2abm)u+2mbc+2ab^2}{m^3}+v^2. \tag{3.10}$$

In fact, (3.3) becomes

$$\begin{cases}\dot{u}=v,\\ \dot{v}=-\frac{m}{2b}v^2+\left(\frac{c}{b}u-\frac{a}{2b}u^2\right).\end{cases} \tag{3.11}$$

According to the Weierstrass inverse integrating factor (3.10), it is easy to know the solution of (3.11) is

$$-\frac{1}{2}\ln|v^2+\frac{am^2u^2-(2m^2c+2abm)u+2mbc+2ab^2}{m^3}|-\frac{m}{2b}u=c_5,$$

where $u'=v$ and c_5 is an arbitrary constant. Then, we can obtain an equation satisfied by traveling wave solutions.

$$u'^2=-\frac{a}{m}u^2+\frac{2cm+2ab}{m^2}u-\frac{2cbm+2ab^2}{m^3}+c_6e^{-\frac{m}{b}u}.$$

Let $c_6=0$, that is

$$u'=\pm\sqrt{-\frac{a}{m}u^2+\frac{2cm+2ab}{m^2}u-\frac{2cbm+2ab^2}{m^3}}.$$

Next, we will obtain the solutions of it in two cases.

(a)When $\frac{a}{m}>0$, the exact solution of the corresponding (3.1) is

$$u=\frac{\sqrt{c^2m^4-4abcm^3-a^2b^2m^2}\sin\left(c_7\sqrt{\frac{a}{m}}\pm\sqrt{\frac{a}{m}}\xi\right)-abm+cm^2}{am^2},$$

that is

$$u=\frac{\sqrt{c^2m^4-4abcm^3-a^2b^2m^2}\sin\left(c_7\sqrt{\frac{a}{m}}\pm\sqrt{\frac{a}{m}}(x-ct)\right)-abm+cm^2}{am^2},$$

where c_7 is an arbitrary constant.

(b)When $\frac{a}{m}<0$, the exact solution of the corresponding (3.1) is

$$\begin{aligned}&u-\frac{m}{a}\sqrt{\frac{a^2m^2u^2-(2acm^2-2a^2bm)\,u+2abcm+2a^2b^2}{m^4}}\\&=-\frac{m}{a}c_8e^{\xi}+\frac{cm-ab}{am},\end{aligned}$$

that is

$$\begin{aligned}&u-\frac{m}{a}\sqrt{\frac{a^2m^2u^2-(2acm^2-2a^2bm)\,u+2abcm+2a^2b^2}{m^4}}\\&=-\frac{m}{a}c_8e^{x-ct}+\frac{cm-ab}{am},\end{aligned}$$

where c_8 is an arbitrary constant.

(5) When $s = 2$ and $f_1(u)f_0(u) \neq 0$, we have $n = 0$ and the Weierstrass inverse integrating factor is

$$\mu(u,v) = \frac{E(u,v)}{-amu^3 + (2mc - 2ba)u^2}, \tag{3.12}$$

where

$$\begin{aligned} E(u,v) = &-a^2u^5 + 4acu^3 - 4c^2u^2 + (4fcu^3 - 2afu^4)v \\ &+ (2mcu^2 - 2bau^2 - abu^3)v^2. \end{aligned}$$

In fact, (3.3) becomes

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -\frac{m}{2b}v^2 - \left(\frac{f}{b}u\right)v + \left(\frac{c}{b}u - \frac{a}{2b}u^2\right). \end{cases} \tag{3.13}$$

According to the Weierstrass inverse integration factor (3.12), it is easy to know the solution of (3.13) is

$$\begin{aligned} &\frac{1}{2}\ln\left|\frac{F(u,v)}{-amu^3 + (2mc - 2ba)u^2}\right| \\ &- \frac{4fcu^3 - 2afu^4}{\sqrt{G(u,v)}} \arctan\frac{(4mcv - 4abv)u^2 + (4fc - 2amv)u^3 - 2afu^4}{\sqrt{G(u,v)}} \\ &- \int_0^u \frac{a^2mu^5 + \left(2a^2b - 4amc\right)u^4 + \left(4mc^2 - 4abc\right)u^3}{-2a^2bu^5 + \left(8abc - 8bc^2\right)u^3} du = c_9, \end{aligned}$$

where c_9 is an arbitrary constant and

$$\begin{aligned} F(u,v) = &(4acu^3 - 4c^2u^3 - a^2u^5) + (4fcu^3 - 2afu^4)v \\ &+ (2mcu^2 - 2abu^2 - amu^3)v^2, \\ G(u,v) = &4(2mcu^2 - 2abu^2 - amu^3)(4acu^3 - 4c^2u^3 - a^2u^5) \\ &- (4fcu^3 - 2afu^4)^2. \end{aligned}$$

Because $u' = v$, we get an equation satisfied by traveling wave solution.

In summary, according to the Weierstrass inverse integration factor, we obtain the traveling wave solutions or the equation satisfied by the traveling wave solution of (3.1).

4 Conclusions

In this paper, we consider the generalized Weierstrass integrability of a class of second order nonlinear differential equations by presenting the existence conditions and the expressions of generalized Weierstrass inverse integrating factors. The relationship between the generalized Weierstrass inverse integrating factor and the Weierstrass inverse integrating factor is also presented. At last, an application is considered, the traveling wave solutions or the equation satisfied by the traveling wave solutions of a Kudryashov-Sinelshchikov equation are obtained, which will be helpful for further research on the Kudryashov-Sinelshchikov equation.

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Competing Interests

Authors have declared that no competing interests exist.

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