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A Study on Sum Formulas of Generalized Hexanacci Numbers: Closed Forms of the Sum Formulas $\sum_{k=0}^{n} x^k W_k$ and $\sum_{k=1}^{n} x^k W_{-k}$

Yüksel Soykan[1*]

[1] *Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} x^k W_k$ and $\sum_{k=1}^{n} x^k W_{-k}$ for generalized Hexanacci numbers are presented. As special cases, we give summation formulas of Hexanacci, Hexanacci-Lucas, and other sixth-order recurrence sequences.

Keywords: Hexanacci numbers; Hexanacci-Lucas numbers; sum formulas; summing formulas.

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**Corresponding author: E-mail: yuksel_soykan@hotmail.com;*

1 Introduction

The generalized Hexanacci sequence $\{W_n(W_0, W_1, W_2, W_3, W_4, W_5; r, s, t, u, v, y)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$\begin{aligned} W_n &= rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}, \\ W_0 &= c_0, W_1 = c_1, W_2 = c_2,\ W_3 = c_3, W_4 = c_4, W_5 = c_5,\ n \geq 6 \end{aligned} \tag{1.1}$$

where $W_0, W_1, W_2, W_3, W_4, W_5$ are arbitrary real or complex numbers and r, s, t, u, v, y are real numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{v}{y}W_{-n+1} - \frac{u}{y}W_{-n+2} - \frac{t}{y}W_{-n+3} - \frac{s}{y}W_{-n+4} - \frac{r}{y}W_{-n+5} + \frac{1}{y}W_{-n+6}$$

for $n = 1, 2, 3, ...$ when $y \neq 0$. Therefore, recurrence (1.1) holds for all integer n.Hexanacci sequence has been studied by many authors, see for example [1,2,3] and references therein.

Table 1. A few special case of generalized Hexanacci sequences

No	Sequences (Numbers)	Notation	References
1	Generalized Hexanacci	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 1, 1, 1, 1, 1, 1)\}$	[4]
2	Generalized Sixth order Pell	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 2, 1, 1, 1, 1, 1)\}$	[5]
3	Generalized Sixth order Jacobsthal	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 1, 1, 1, 1, 1, 2)\}$	[6]
4	Generalized 6-primes	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 2, 3, 5, 7, 11, 13)\}$	[7]

For some specific values of $W_0, W_1, W_2, W_3, W_4, W_5$ and r, s, t, u, v, y it is worth presenting these special Hexanacci numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 2) are used for the special cases of r, s, t, u, v, y and initial values.

Table 2. A few members of generalized Hexanacci sequences

Sequences (Numbers)	Notation	OEIS [8]	Ref
Hexanacci	$\{H_n\} = \{W_n(0, 1, 1, 2, 4, 8; 1, 1, 1, 1, 1, 1)\}$	A001592	[4]
Hexanacci-Lucas	$\{E_n\} = \{W_n(6, 1, 3, 7, 15, 31; 1, 1, 1, 1, 1, 1)\}$	A074584	[4]
sixth order Pell	$\{P_n^{(6)}\} = \{W_n(0, 1, 2, 5, 13, 34; 2, 1, 1, 1, 1, 1)\}$		[5]
sixth order Pell-Lucas	$\{Q_n^{(6)}\} = \{W_n(6, 2, 6, 17, 46, 122; 2, 1, 1, 1, 1, 1)\}$		[5]
modified sixth order Pell	$\{E_n^{(6)}\} = \{W_n(0, 1, 1, 3, 8, 21; 2, 1, 1, 1, 1, 1)\}$		[5]
sixth order Jacobsthal	$\{J_n^{(6)}\} = \{W_n(0, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 2)\}$		[6,9]
sixth order Jacobsthal-Lucas	$\{j_n^{(6)}\} = \{W_n(2, 1, 5, 10, 20, 40; 1, 1, 1, 1, 1, 2)\}$		[6,9]
modified sixth order Jacobsthal	$\{K_n^{(6)}\} = \{W_n(3, 1, 3, 10, 20, 40; 1, 1, 1, 1, 1, 2)\}$		[6]
sixth-order Jacobsthal Perrin	$\{Q_n^{(6)}\} = \{W_n(3, 0, 2, 8, 16, 32; 1, 1, 1, 1, 1, 2)\}$		[6]
adjusted sixth-order Jacobsthal	$\{S_n^{(6)}\} = \{W_n(0, 1, 1, 2, 4, 8; 1, 1, 1, 1, 1, 2)\}$		[6]
modified sixth-order Jacobsthal-Lucas	$\{R_n^{(6)}\} = \{W_n(6, 1, 3, 7, 15, 31; 1, 1, 1, 1, 1, 2)\}$		[6]
6-primes	$\{G_n\} = \{W_n(0, 0, 0, 0, 1, 2; 2, 3, 5, 7, 11, 13)\}$		[7]
Lucas 6-primes	$\{H_n\} = \{W_n(6, 2, 10, 41, 150, 542; 2, 3, 5, 7, 11, 13)\}$		[7]
modified 6-primes	$\{E_n\} = \{W_n(0, 0, 0, 0, 1, 1; 2, 3, 5, 7, 11, 13)\}$		[7]

For easy writing, from now on, we drop the superscripts from the sequences, for example we write P_n for $P_n^{(6)}$.

We present some works on summing formulas of the numbers in the following Table 3.

Table 3. A few special study of sum formulas

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[10,11,12],[13,14]
Generalized Fibonacci	[15,16,17,18,19,20,21]
Generalized Tribonacci	[22,23,24]
Generalized Tetranacci	[25,26,27]
Generalized Pentanacci	[28,29]
Generalized Hexanacci	[30,31]

In this work, we investigate summation formulas of generalized Hexanacci numbers.

2 Sum Formulas of Generalized Hexanacci Numbers with Positive Subscripts

The following theorem presents some summing formulas of generalized Hexanacci numbers with positive subscripts.

Theorem 2.1. *Let x be a real (or complex) number. For $n \geq 0$ we have the following formulas:*

(a) *If $sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_k = \frac{\Theta_1(x)}{\Theta(x)}$$

where

$\Theta_1(x) = x^{n+5}W_{n+5} - (rx-1)x^{n+4}W_{n+4} - (sx^2+rx-1)x^{n+3}W_{n+3} - (sx^2+tx^3+rx-1)x^{n+2}W_{n+2} - (sx^2+tx^3+ux^4+rx-1)x^{n+1}W_{n+1} + yx^{n+6}W_n - x^5W_5 + x^4(rx-1)W_4 + x^3(sx^2+rx-1)W_3 + x^2(sx^2+tx^3+rx-1)W_2 + x(sx^2+tx^3+ux^4+rx-1)W_1 + (sx^2+tx^3+ux^4+vx^5+rx-1)W_0$

and

$\Theta(x) = sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1.$

(b) *If $r^2x + 2ux^2 + 2x^3y - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 - x^6y^2 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 2sx^4y - 2ux^5y - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{\Theta_2(x)}{\Delta_1}$$

where

$\Theta_2(x) = -\left(ux^2 + x^3y + sx - 1\right)x^{n+1}W_{2n+2} + (t + rs + vx + rx^2y + rux)x^{n+2}W_{2n+1} + (u + t^2x - u^2x^2 + v^2x^3 - x^4y^2 + rt + xy + 2tvx^2 - sx^2y - 2ux^3y + rvx - sux)x^{n+2}W_{2n} + (v + ru + tx^2y - svx + tux + rxy)x^{n+2}W_{2n-1} + (y + v^2x^2 - x^3y^2 + rv - ux^2y + tvx - sxy)x^{n+2}W_{2n-2} + y(r + vx^2 + tx)x^{n+2}W_{2n-3} - x^3(r + vx^2 + tx)W_5 + x^2(r^2x + ux^2 + x^3y + sx + rtx^2 + rvx^3 - 1)W_4 - x^3(t + vx - svx^2 + rx^2y + rux - stx)W_3 + x(r^2x + ux^2 + x^3y - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 + rvx^3 - sux^3 + tvx^4 - sx^4y - 1)W_2 - x^3(v - uvx^2 + tx^2y - svx + rxy)W_1 + (r^2x + 2ux^2 + x^3y - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - sx^4y - ux^5y - 1)W_0,$

and

$\Delta_1 = r^2x + 2ux^2 + 2x^3y - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 - x^6y^2 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 2sx^4y - 2ux^5y - 1.$

(c) *If* $r^2x+2ux^2+2x^3y-s^2x^2+t^2x^3-u^2x^4+v^2x^5-x^6y^2+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-2sx^4y-2ux^5y-1=0$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{\Theta_3(x)}{\Delta_1}$$

where

$\Theta_3(x) = (r+vx^2+tx)x^{n+1}W_{2n+2} + (s-s^2x+x^2y+t^2x^2-u^2x^3+v^2x^4-x^5y^2+ux+rvx^2-2sux^2+2tvx^3-2sx^3y-2ux^4y+rtx)x^{n+1}W_{2n+1} + (t+vx-svx^2+rx^2y+rux-stx)x^{n+1}W_{2n} + (u-u^2x^2+v^2x^3-x^4y^2+xy+tvx^2-sx^2y-2ux^3y+rvx-sux)x^{n+1}W_{2n-1} + (v-uvx^2+tx^2y-svx+rxy)x^{n+1}W_{2n-2} - yx^{n+1}(ux^2+x^3y+sx-1)W_{2n-3} + x^2(ux^2+x^3y+sx-1)W_5 - x^3(t+rs+vx+rx^2y+rux)W_4 + x(r^2x+ux^2+x^3y-s^2x^2+2sx+rtx^2+rvx^3-sux^3-sx^4y-1)W_3 - x^3\left(v+ru+tx^2y-svx+tux+rxy\right)W_2 + (r^2x+2ux^2+x^3y-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2+rvx^3-2sux^3+tvx^4-sx^4y-ux^5y-1)W_1 - x^3y(r+vx^2+tx)W_0.$

Proof.

(a) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}$$

i.e.

$$yW_{n-6} = W_n - rW_{n-1} - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5}$$

we obtain

$$\begin{array}{rcl}
yx^0W_0 &=& x^0W_6 - rx^0W_5 - sx^0W_4 - tx^0W_3 - ux^0W_2 - vx^0W_1 \\
yx^1W_1 &=& x^1W_7 - rx^1W_6 - sx^1W_5 - tx^1W_4 - ux^1W_3 - vx^1W_2 \\
yx^2W_2 &=& x^2W_8 - rx^2W_7 - sx^2W_6 - tx^2W_5 - ux^2W_4 - vx^2W_3 \\
yx^3W_3 &=& x^3W_9 - rx^3W_8 - sx^3W_7 - tx^3W_6 - ux^3W_5 - vx^3W_4 \\
&\vdots& \\
yx^{n-4}W_{n-4} &=& x^{n-4}W_{n+2} - rx^{n-4}W_{n+1} - sx^{n-4}W_n - tx^{n-4}W_{n-1} - ux^{n-4}W_{n-2} - vx^{n-4}W_{n-3} \\
yx^{n-3}W_{n-3} &=& x^{n-3}W_{n+3} - rx^{n-3}W_{n+2} - sx^{n-3}W_{n+1} - tx^{n-3}W_n - ux^{n-3}W_{n-1} - vx^{n-3}W_{n-2} \\
yx^{n-2}W_{n-2} &=& x^{n-2}W_{n+4} - rx^{n-2}W_{n+3} - sx^{n-2}W_{n+2} - tx^{n-2}W_{n+1} - ux^{n-2}W_n - vx^{n-2}W_{n-1} \\
yx^{n-1}W_{n-1} &=& x^{n-1}W_{n+5} - rx^{n-1}W_{n+4} - sx^{n-1}W_{n+3} - tx^{n-1}W_{n+2} - ux^{n-1}W_{n+1} - vx^{n-1}W_n \\
yx^nW_n &=& x^nW_{n+6} - rx^nW_{n+5} - sx^nW_{n+4} - tx^nW_{n+3} - ux^nW_{n+2} - vx^nW_{n+1}
\end{array}$$

If we add the equations side by side (and using $W_{n+6} = rW_{n+5} + sW_{n+4} + tW_{n+3} + uW_{n+2} + vW_{n+1} + yW_n$, , we get (a).

Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5} - yW_{n-6}$$

we obtain

$$\begin{array}{rcl}
rx^1W_3 &=& x^1W_4 - sx^1W_2 - tx^1W_1 - ux^1W_0 - vx^1W_{-1} - yx^1W_{-2} \\
rx^2W_5 &=& x^2W_6 - sx^2W_4 - tx^2W_3 - ux^2W_2 - vx^2W_1 - yW_0 \\
rx^3W_7 &=& x^3W_8 - sx^3W_6 - tx^3W_5 - ux^3W_4 - vx^3W_3 - yx^3W_2 \\
rx^4W_9 &=& x^4W_{10} - sx^4W_8 - tx^4W_7 - ux^4W_6 - vx^4W_5 - yx^4W_4 \\
&\vdots& \\
rx^{n-1}W_{2n-1} &=& x^{n-1}W_{2n} - sx^{n-1}W_{2n-2} - tx^{n-1}W_{2n-3} - ux^{n-1}W_{2n-4} - vx^{n-1}W_{2n-5} - yx^{n-1}W_{2n-6} \\
rx^nW_{2n+1} &=& x^nW_{2n+2} - sx^nW_{2n} - tx^nW_{2n-1} - ux^nW_{2n-2} - vx^nW_{2n-3} - yx^nW_{2n-4}
\end{array}$$

Now, if we add the above equations side by side, we get

$$\begin{aligned} r(-x^0W_1+\sum_{k=0}^{n}x^kW_{2k+1}) &= (x^nW_{2n+2}-x^0W_2-x^{-1}W_0+\sum_{k=0}^{n}x^{k-1}W_{2k}) \\ &\quad -s(-x^0W_0+\sum_{k=0}^{n}x^kW_{2k})-t(-x^{n+1}W_{2n+1}+\sum_{k=0}^{n}x^{k+1}W_{2k+1}) \\ &\quad -u(-x^{n+1}W_{2n}+\sum_{k=0}^{n}x^{k+1}W_{2k})-v(-x^{n+2}W_{2n+1}-x^{n+1}W_{2n-1} \\ &\quad +x^1W_{-1}+\sum_{k=0}^{n}x^{k+2}W_{2k+1}) \\ &\quad -y(-x^{n+2}W_{2n}-x^{n+1}W_{2n-2}+x^1W_{-2}+\sum_{k=0}^{n}x^{k+2}W_{2k}) \end{aligned} \tag{2.1}$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}+vW_{n-5}+yW_{n-6}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2}-tW_{n-3}-uW_{n-4}-vW_{n-5}-yW_{n-6}$$

we write the following obvious equations;

$$\begin{aligned} rx^1W_2 &= x^1W_3-sx^1W_1-tx^1W_0-ux^1W_{-1}-vx^1W_{-2}-yx^1W_{-3} \\ rx^2W_4 &= x^2W_5-sx^2W_3-tx^2W_2-ux^2W_1-vx^2W_0-yx^2W_{-1} \\ rx^3W_6 &= x^3W_7-sx^3W_5-tx^3W_4-ux^3W_3-vx^3W_2-yx^3W_1 \\ &\vdots \\ rx^{n-1}W_{2n-2} &= x^{n-1}W_{2n-1}-sx^{n-1}W_{2n-3}-tx^{n-1}W_{2n-4}-ux^{n-1}W_{2n-5}-vx^{n-1}W_{2n-6}-yx^{n-1}W_{2n-7} \\ rx^nW_{2n} &= x^nW_{2n+1}-sx^nW_{2n-1}-tx^nW_{2n-2}-ux^nW_{2n-3}-vx^nW_{2n-4}-yx^nW_{2n-5} \end{aligned}$$

Now, if we add the above equations side by side, we obtain

$$\begin{aligned} r(-x^0W_0+\sum_{k=0}^{n}x^kW_{2k}) &= (-x^0W_1+\sum_{k=0}^{n}x^kW_{2k+1})-s(-x^{n+1}W_{2n+1}+\sum_{k=0}^{n}x^{k+1}W_{2k+1}) \\ &\quad -t(-x^{n+1}W_{2n}+\sum_{k=0}^{n}x^{k+1}W_{2k}) \\ &\quad -u(-x^{n+2}W_{2n+1}-x^{n+1}W_{2n-1}+x^1W_{-1}+\sum_{k=0}^{n}x^{k+2}W_{2k+1}) \\ &\quad -v(-x^{n+2}W_{2n}-x^{n+1}W_{2n-2}+x^1W_{-2}+\sum_{k=0}^{n}x^{k+2}W_{2k}) \\ &\quad -y(-x^{n+3}W_{2n+1}-x^{n+2}W_{2n-1}-x^{n+1}W_{2n-3} \\ &\quad +x^1W_{-3}+x^2W_{-1}+\sum_{k=0}^{n}x^{k+3}W_{2k+1}) \end{aligned} \tag{2.2}$$

Then, solving the system (2.1)-(2.2), the required result of (b) and (c) follow. □

3 Special Cases

In this section, for the special cases of x, we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n}x^kW_k$, $\sum_{k=0}^{n}x^kW_{2k}$ and $\sum_{k=0}^{n}x^kW_{2k+1}$ for the specific case of sequence $\{W_n\}$.

3.1 The case $x = 1$

In this subsection we consider the special case $x = 1$.

The case $x = 1$ of Theorem 2.1 is given in Soykan [31, Theorem 2.1]. For the generalized 6-primes sequence case $(x = 1, r = 2, s = 3, t = 5, u = 7, v = 11, y = 13)$, see [7].

We only consider the case $x = 1, r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ (which is not considered in [31]).

Observe that setting $x = 1, r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ (i.e. for the generalized sixth order Jacobsthal sequence case) in Theorem 2.1 (b), (c) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

Theorem 3.1. *If $r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{6}(W_{n+5} - W_{n+3} - 2W_{n+2} - 3W_{n+1} + 2W_n - W_5 + W_3 + 2W_2 + 3W_1 + 4W_0)$.

(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{9}((n+4)W_{2n+2} - 2(n+3)W_{2n+1} + (10+n)W_{2n} - 2(n+3)W_{2n-1} + (n+7)W_{2n-2} - 2(n+3)W_{2n-3} + 4W_5 - 9W_4 + 4W_3 - 6W_2 + 4W_1 - 3W_0)$.

(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{9}(-(n+2)W_{2n+2} + 2(n+7)W_{2n+1} - (n+2)W_{2n} + (2n+11)W_{2n-1} - (n+2)W_{2n-2} + 2(n+4)W_{2n-3} - 5W_5 + 8W_4 - 2W_3 + 8W_2 + W_1 + 8W_0)$.

Proof.

(a) We use Theorem 2.1 (a). If we set $x = 1, r = 1, s = 1, t = 1, u = 2$ in Theorem 2.1 (a) we get (a).

(b) We use Theorem 2.1 (b). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 2.1 (b) then we have

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{g_1(x)}{-(4x-1)(x-1)(x+x^2+1)^2}$$

where
$g_1(x) = -x^{n+1}(2x^3 + x^2 + x - 1)W_{2n+2} + x^{n+2}(2x^2 + 2x + 2)W_{2n+1} - x^{n+2}(4x^4 + 3x^3 + x^2 - 3x - 2)W_{2n} + x^{n+2}(2x^2 + 2x + 2)W_{2n-1} - x^{n+2}(4x^3 + x^2 + x - 3)W_{2n-2} + 2x^{n+2}(x^2 + x + 1)W_{2n-3} - x^3(x^2 + x + 1)W_5 + x^2(3x^3 + 2x^2 + 2x - 1)W_4 - x^3(x^2 + x + 1)W_3 + x(-x^4 + 3x^3 + 2x^2 + 3x - 1)W_2 - x^3(x^2 + x + 1)W_1 + (-x^5 - x^4 + 3x^3 + 3x^2 + 3x - 1)W_0$.
For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) using

$$\begin{aligned}\sum_{k=0}^{n} W_{2k} &= \left.\frac{\frac{d}{dx}(g_1(x))}{\frac{d}{dx}(-(4x-1)(x-1)(x+x^2+1)^2)}\right|_{x=1} \\ &= \frac{1}{9}((n+4)W_{2n+2} - 2(n+3)W_{2n+1} + (10+n)W_{2n} - 2(n+3)W_{2n-1} \\ &\quad +(n+7)W_{2n-2} - 2(n+3)W_{2n-3} + 4W_5 - 9W_4 + 4W_3 - 6W_2 + 4W_1 - 3W_0).\end{aligned}$$

(c) We use Theorem 2.1 (c). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 2.1 (c) then we have

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{g_2(x)}{-(4x-1)(x-1)(x+x^2+1)^2}$$

where
$g_2(x) = x^{n+1}(x^2 + x + 1)W_{2n+2} + x^{n+1}(-4x^5 - 3x^4 - 3x^3 + 2x^2 + x + 1)W_{2n+1} + x^{n+1}(x^2 + x + 1)W_{2n} - x^{n+1}(4x^4 + 3x^3 + 2x^2 - 2x - 1)W_{2n-1} + x^{n+1}(x^2 + x + 1)W_{2n-2} - 2x^{n+1}$

$(2x^3+x^2+x-1)W_{2n-3}+x^2(2x^3+x^2+x-1)W_5-x^3(2x^2+2x+2)W_4+x(-2x^4+2x^3+x^2+3x-1)W_3-x^3(2x^2+2x+2)W_2+(-2x^5-2x^4+2x^3+3x^2+3x-1)W_1-2x^3(x^2+x+1)W_0$.

For $x=1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get (c) using

$$\begin{aligned}\sum_{k=0}^{n} W_{2k+1} &= \left.\frac{\frac{d}{dx}\left(g_2(x)\right)}{\frac{d}{dx}\left(-(4x-1)(x-1)(x+x^2+1)^2\right)}\right|_{x=1} \\ &= \frac{1}{9}(-(n+2)W_{2n+2}+2(n+7)W_{2n+1}-(n+2)W_{2n}+(2n+11)W_{2n-1} \\ &\quad -(n+2)W_{2n-2}+2(n+4)W_{2n-3}-5W_5+8W_4-2W_3+8W_2+W_1+8W_0).\end{aligned}$$

□

Taking $W_n=J_n$ with $J_0=0, J_1=1, J_2=1, J_3=1, J_4=1, J_5=1$ in the last Theorem, we have the following Corollary which presents sum formulas of sixth-order Jacobsthal numbers.

Corollary 3.2. *For $n \geq 0$, sixth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} J_k = \frac{1}{6}(J_{n+5}-J_{n+3}-2J_{n+2}-3J_{n+1}+2J_n+5)$.

(b) $\sum_{k=0}^{n} J_{2k} = \frac{1}{9}((n+4)J_{2n+2}-2(n+3)J_{2n+1}+(10+n)J_{2n}-2(n+3)J_{2n-1}+(n+7)J_{2n-2}-2(n+3)J_{2n-3}-3)$.

(c) $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{9}(-(n+2)J_{2n+2}+2(n+7)J_{2n+1}-(n+2)J_{2n}+(2n+11)J_{2n-1}-(n+2)J_{2n-2}+2(n+4)J_{2n-3}+10)$.

From the last Theorem, we have the following Corollary which gives sum formulas of sixth order Jacobsthal-Lucas numbers (take $W_n=j_n$ with $j_0=2, j_1=1, j_2=5, j_3=10, j_4=20, j_5=40$).

Corollary 3.3. *For $n \geq 0$, sixth order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} j_k = \frac{1}{6}(j_{n+5}-j_{n+3}-2j_{n+2}-3j_{n+1}+2j_n-9)$.

(b) $\sum_{k=0}^{n} j_{2k} = \frac{1}{9}((n+4)j_{2n+2}-2(n+3)j_{2n+1}+(10+n)j_{2n}-2(n+3)j_{2n-1}+(n+7)j_{2n-2}-2(n+3)j_{2n-3}-12)$.

(c) $\sum_{k=0}^{n} j_{2k+1} = \frac{1}{9}(-(n+2)j_{2n+2}+2(n+7)j_{2n+1}-(n+2)j_{2n}+(2n+11)j_{2n-1}-(n+2)j_{2n-2}+2(n+4)j_{2n-3}-3)$.

Taking $W_n=K_n$ with $K_0=3, K_1=1, K_2=3, K_3=10, K_4=20, K_5=40$ in the last Theorem, we have the following corollary which presents sum formula of modified sixth order Jacobsthal numbers.

Corollary 3.4. *For $n \geq 0$,modified sixth order Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n} K_k = \frac{1}{6}(K_{n+5}-K_{n+3}-2K_{n+2}-3K_{n+1}+2K_n-9)$.

(b) $\sum_{k=0}^{n} K_{2k} = \frac{1}{9}((n+4)K_{2n+2}-2(n+3)K_{2n+1}+(10+n)K_{2n}-2(n+3)K_{2n-1}+(n+7)K_{2n-2}-2(n+3)K_{2n-3}-3)$.

(c) $\sum_{k=0}^{n} K_{2k+1} = \frac{1}{9}(-(n+2)K_{2n+2}+2(n+7)K_{2n+1}-(n+2)K_{2n}+(2n+11)K_{2n-1}-(n+2)K_{2n-2}+2(n+4)K_{2n-3}-11)$.

From the last Theorem, we have the following corollary which gives sum formula of sixth-order Jacobsthal Perrin numbers (take $W_n=Q_n$ with $Q_0=3, Q_1=0, Q_2=2, Q_3=8, Q_4=16, Q_5=32$).

Corollary 3.5. *For $n \geq 0$, sixth-order Jacobsthal Perrin numbers have the following property:*

(a) $\sum_{k=0}^{n} Q_k = \frac{1}{6}(Q_{n+5}-Q_{n+3}-2Q_{n+2}-3Q_{n+1}+2Q_n-8)$.

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{9}((n+4)Q_{2n+2} - 2(n+3)Q_{2n+1} + (10+n)Q_{2n} - 2(n+3)Q_{2n-1} + (n+7)Q_{2n-2} - 2(n+3)Q_{2n-3} - 5)$.

(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{9}(-(n+2)Q_{2n+2} + 2(n+7)Q_{2n+1} - (n+2)Q_{2n} + (2n+11)Q_{2n-1} - (n+2)Q_{2n-2} + 2(n+4)Q_{2n-3} - 8)$.

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4, S_5 = 8$ in the Theorem, we have the following corollary which presents sum formula of adjusted sixth-order Jacobsthal numbers.

Corollary 3.6. *For $n \geq 0$, adjusted sixth-order Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n} S_k = \frac{1}{6}(S_{n+5} - S_{n+3} - 2S_{n+2} - 3S_{n+1} + 2S_n - 1)$.

(b) $\sum_{k=0}^{n} S_{2k} = \frac{1}{9}((n+4)S_{2n+2} - 2(n+3)S_{2n+1} + (10+n)S_{2n} - 2(n+3)S_{2n-1} + (n+7)S_{2n-2} - 2(n+3)S_{2n-3} + 2)$.

(c) $\sum_{k=0}^{n} S_{2k+1} = \frac{1}{9}(-(n+2)S_{2n+2} + 2(n+7)S_{2n+1} - (n+2)S_{2n} + (2n+11)S_{2n-1} - (n+2)S_{2n-2} + 2(n+4)S_{2n-3} - 3)$.

From the last Theorem, we have the following corollary which gives sum formula of modified sixth-order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 6, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15, R_5 = 31$).

Corollary 3.7. *For $n \geq 0$, modified sixth-order Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=0}^{n} R_k = \frac{1}{6}(R_{n+5} - R_{n+3} - 2R_{n+2} - 3R_{n+1} + 2R_n + 9)$.

(b) $\sum_{k=0}^{n} R_{2k} = \frac{1}{9}((n+4)R_{2n+2} - 2(n+3)R_{2n+1} + (10+n)R_{2n} - 2(n+3)R_{2n-1} + (n+7)R_{2n-2} - 2(n+3)R_{2n-3} - 15)$.

(c) $\sum_{k=0}^{n} R_{2k+1} = \frac{1}{9}(-(n+2)R_{2n+2} + 2(n+7)R_{2n+1} - (n+2)R_{2n} + (2n+11)R_{2n-1} - (n+2)R_{2n-2} + 2(n+4)R_{2n-3} + 24)$.

3.2 The case $x = -1$

In this subsection we consider the special case $x = -1$ and we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n}(-1)^k W_k$, $\sum_{k=0}^{n}(-1)^k W_{2k}$ and $\sum_{k=0}^{n}(-1)^k W_{2k+1}$ for the specific case of the sequence $\{W_n\}$.

Taking $r = s = t = u = v = y = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 3.1. *If $r = s = t = u = v = y = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k = (-1)^n (W_{n+5} - 2W_{n+4} + W_{n+3} - 2W_{n+2} + W_{n+1} - W_n) - W_5 + 2W_4 - W_3 + 2W_2 - W_1 + 2W_0$.

(b) $\sum_{k=0}^{n}(-1)^k W_{2k} = \frac{1}{5}((-1)^n (2W_{2n+2} - W_{2n+1} - 2W_{2n-1} - 3W_{2n-2} - W_{2n-3}) - W_5 + 3W_4 - 4W_2 - W_1 + 3W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{2k+1} = \frac{1}{5}((-1)^n (W_{2n+2} + 2W_{2n+1} - W_{2n-1} + W_{2n-2} + 2W_{2n-3}) + 2W_5 - W_4 - 5W_3 - 2W_2 + 2W_1 - W_0)$.

From the above Proposition, we have the following Corollary which gives sum formulas of Hexanacci numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$).

Corollary 3.8. *For $n \geq 0$, Hexanacci numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k H_k = (-1)^n (H_{n+5} - 2H_{n+4} + H_{n+3} - 2H_{n+2} + H_{n+1} - H_n) - 1.$

(b) $\sum_{k=0}^{n}(-1)^k H_{2k} = \frac{1}{5}((-1)^n (2H_{2n+2} - H_{2n+1} - 2H_{2n-1} - 3H_{2n-2} - H_{2n-3}) - 1).$

(c) $\sum_{k=0}^{n}(-1)^k H_{2k+1} = \frac{1}{5}((-1)^n (H_{2n+2} + 2H_{2n+1} - H_{2n-1} + H_{2n-2} + 2H_{2n-3}) + 2).$

Taking $W_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$ in the above Proposition, we have the following Corollary which presents sum formulas of Hexanacci-Lucas numbers.

Corollary 3.9. *For $n \geq 0$, Hexanacci-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k E_k = (-1)^n (E_{n+5} - 2E_{n+4} + E_{n+3} - 2E_{n+2} + E_{n+1} - E_n) + 9.$

(b) $\sum_{k=0}^{n}(-1)^k E_{2k} = \frac{1}{5}((-1)^n (2E_{2n+2} - E_{2n+1} - 2E_{2n-1} - 3E_{2n-2} - E_{2n-3}) + 19).$

(c) $\sum_{k=0}^{n}(-1)^k E_{2k+1} = \frac{1}{5}((-1)^n (E_{2n+2} + 2E_{2n+1} - E_{2n-1} + E_{2n-2} + 2E_{2n-3}) + 2).$

Taking $r = 2, s = t = u = v = y = 1$ in Theorem 2.1 (a), (b) and (c), we obtain the following Proposition.

Proposition 3.2. *If $r = 2, s = t = u = v = y = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k = \frac{1}{2}((-1)^n (W_{n+5} - 3W_{n+4} + 2W_{n+3} - 3W_{n+2} + 2W_{n+1} - W_n) - W_5 + 3W_4 - 2W_3 + 3W_2 - 2W_1 + 3W_0).$

(b) $\sum_{k=0}^{n}(-1)^k W_{2k} = \frac{1}{4}((-1)^n (W_{2n+2} - W_{2n+1} - W_{2n-1} - 2W_{2n-2} - W_{2n-3}) - W_5 + 3W_4 - 3W_2 + 3W_0).$

(c) $\sum_{k=0}^{n}(-1)^k W_{2k+1} = \frac{1}{4}((-1)^n (W_{2n+2} + W_{2n+1} - W_{2n-1} + W_{2n-3}) + W_5 - W_4 - 4W_3 - W_2 + 2W_1 - W_0).$

From the last Proposition, we have the following Corollary which gives sum formulas of sixth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13, P_5 = 34$).

Corollary 3.10. *For $n \geq 0$, sixth-order Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k P_k = \frac{1}{2}((-1)^n (P_{n+5} - 3P_{n+4} + 2P_{n+3} - 3P_{n+2} + 2P_{n+1} - P_n) - 1).$

(b) $\sum_{k=0}^{n}(-1)^k P_{2k} = \frac{1}{4}((-1)^n (P_{2n+2} - P_{2n+1} - P_{2n-1} - 2P_{2n-2} - P_{2n-3}) - 1).$

(c) $\sum_{k=0}^{n}(-1)^k P_{2k+1} = \frac{1}{4}((-1)^n (P_{2n+2} + P_{2n+1} - P_{2n-1} + P_{2n-3}) + 1).$

Taking $W_n = Q_n$ with $Q_0 = 6, Q_1 = 2, Q_2 = 6, Q_3 = 17, Q_4 = 46, Q_5 = 122$ in the last Proposition, we have the following Corollary which presents sum formulas of sixth-order Pell-Lucas numbers.

Corollary 3.11. *For $n \geq 0$, sixth-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k Q_k = \frac{1}{2}((-1)^n (Q_{n+5} - 3Q_{n+4} + 2Q_{n+3} - 3Q_{n+2} + 2Q_{n+1} - Q_n) + 14).$

(b) $\sum_{k=0}^{n}(-1)^k Q_{2k} = \frac{1}{4}((-1)^n (Q_{2n+2} - Q_{2n+1} - Q_{2n-1} - 2Q_{2n-2} - Q_{2n-3}) + 16).$

(c) $\sum_{k=0}^{n}(-1)^k Q_{2k+1} = \frac{1}{4}((-1)^n (Q_{2n+2} + Q_{2n+1} - Q_{2n-1} + Q_{2n-3})).$

Observe that setting $x = -1, r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ (i.e. for the generalized sixth order Jacobsthal case) in Theorem 2.1 (a), makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

Theorem 3.12. *If $r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k = \frac{1}{9}(-(-1)^n(n+5)W_{n+5} + (-1)^n(2n+9)W_{n+4} - (-1)^n(n+2)W_{n+3} + 2(-1)^n(n+3)W_{n+2} - (-1)^n(n-1)W_{n+1} + 2(-1)^n(n+6)W_n + 5W_5 - 9W_4 + 2W_3 - 6W_2 - W_1 - 3W_0)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{2k} = \frac{1}{10}((-1)^n(3W_{2n+2} - 2W_{2n+1} + 3W_{2n} - 2W_{2n-1} - 7W_{2n-2} - 2W_{2n-3}) - W_5 + 4W_4 - W_3 - 6W_2 - W_1 + 4W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{2k+1} = \frac{1}{10}((-1)^n(W_{2n+2} + 6W_{2n+1} + W_{2n} - 4W_{2n-1} + W_{2n-2} + 6W_{2n-3}) + 3W_5 - 2W_4 - 7W_3 - 2W_2 + 3W_1 - 2W_0)$.

Proof.

(a) We use Theorem 2.1 (a). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 2.1 (a) then we have

$$\sum_{k=0}^{n} x^k W_k = \frac{g_3(x)}{(2x-1)(x+1)(-x+x^2+1)(x+x^2+1)}$$

where
$g_3(x) = x^{n+5}W_{n+5} - x^{n+4}(x-1)W_{n+4} - x^{n+3}(x^2+x-1)W_{n+3} - x^{n+2}(x^3+x^2+x-1)W_{n+2} - x^{n+1}(x^4+x^3+x^2+x-1)W_{n+1} + 2x^{n+6}W_n - x^5W_5 + x^4(x-1)W_4 + x^3(x^2+x-1)W_3 + x^2(x^3+x^2+x-1)W_2 + x(x^4+x^3+x^2+x-1)W_1 + (x^5+x^4+x^3+x^2+x-1)W_0$.
For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) using

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k W_k &= \left.\frac{\frac{d}{dx}(g_3(x))}{\frac{d}{dx}((2x-1)(x+1)(-x+x^2+1)(x+x^2+1))}\right|_{x=-1} \\ &= \frac{1}{9}(-(-1)^n(n+5)W_{n+5} + (-1)^n(2n+9)W_{n+4} \\ &\quad -(-1)^n(n+2)W_{n+3} + 2(-1)^n(n+3)W_{n+2} - (-1)^n(n-1)W_{n+1} \\ &\quad +2(-1)^n(n+6)W_n + 5W_5 - 9W_4 + 2W_3 - 6W_2 - W_1 - 3W_0).\end{aligned}$$

(b) Take $x = -1, r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$ in Theorem 2.1 (b).

(c) Take $x = -1, r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$ in Theorem 2.1 (b). □

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1, J_5 = 1$ in the last Theorem, we have the following Corollary which presents sum formulas of sixth-order Jacobsthal numbers.

Corollary 3.13. *For $n \geq 0$, sixth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k J_k = \frac{1}{9}(-(-1)^n(n+5)J_{n+5} + (-1)^n(2n+9)J_{n+4} - (-1)^n(n+2)J_{n+3} + 2(-1)^n(n+3)J_{n+2} - (-1)^n(n-1)J_{n+1} + 2(-1)^n(n+6)J_n - 9)$.

(b) $\sum_{k=0}^{n}(-1)^k J_{2k} = \frac{1}{10}((-1)^n(3J_{2n+2} - 2J_{2n+1} + 3J_{2n} - 2J_{2n-1} - 7J_{2n-2} - 2J_{2n-3}) - 5)$.

(c) $\sum_{k=0}^{n}(-1)^k J_{2k+1} = \frac{1}{10}((-1)^n(J_{2n+2} + 6J_{2n+1} + J_{2n} - 4J_{2n-1} + J_{2n-2} + 6J_{2n-3}) - 5)$.

From the last Theorem, we have the following Corollary which gives sum formulas of sixth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20, j_5 = 40$).

Corollary 3.14. *For $n \geq 0$, sixth order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k j_k = \frac{1}{9}(-(-1)^n(n+5)j_{n+5} + (-1)^n(2n+9)j_{n+4} - (-1)^n(n+2)j_{n+3} + 2(-1)^n(n+3)j_{n+2} - (-1)^n(n-1)j_{n+1} + 2(-1)^n(n+6)j_n + 3)$.

(b) $\sum_{k=0}^{n}(-1)^k j_{2k} = \frac{1}{10}((-1)^n(3j_{2n+2} - 2j_{2n+1} + 3j_{2n} - 2j_{2n-1} - 7j_{2n-2} - 2j_{2n-3}) + 7)$.

(c) $\sum_{k=0}^{n}(-1)^k j_{2k+1} = \frac{1}{10}((-1)^n(j_{2n+2} + 6j_{2n+1} + j_{2n} - 4j_{2n-1} + j_{2n-2} + 6j_{2n-3}) - 1)$.

Taking $j_n = W_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20, K_5 = 40$ in the last Theorem, we have the following corollary which presents sum formula of modified sixth order Jacobsthal numbers.

Corollary 3.15. *For $n \geq 0$,modified sixth order Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n}(-1)^k K_k = \frac{1}{9}(-(-1)^n(n+5)K_{n+5} + (-1)^n(2n+9)K_{n+4} - (-1)^n(n+2)K_{n+3} + 2(-1)^n(n+3)K_{n+2} - (-1)^n(n-1)K_{n+1} + 2(-1)^n(n+6)K_n + 12)$.

(b) $\sum_{k=0}^{n}(-1)^k K_{2k} = \frac{1}{10}((-1)^n(3K_{2n+2} - 2K_{2n+1} + 3K_{2n} - 2K_{2n-1} - 7K_{2n-2} - 2K_{2n-3}) + 23)$.

(c) $\sum_{k=0}^{n}(-1)^k K_{2k+1} = \frac{1}{10}((-1)^n(K_{2n+2} + 6K_{2n+1} + K_{2n} - 4K_{2n-1} + K_{2n-2} + 6K_{2n-3}) + 1)$.

From the last Theorem, we have the following corollary which gives sum formula of sixth-order Jacobsthal Perrin numbers (take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16, Q_5 = 32$).

Corollary 3.16. *For $n \geq 0$, sixth-order Jacobsthal Perrin numbers have the following property:*

(a) $\sum_{k=0}^{n}(-1)^k Q_k = \frac{1}{9}(-(-1)^n(n+5)Q_{n+5} + (-1)^n(2n+9)Q_{n+4} - (-1)^n(n+2)Q_{n+3} + 2(-1)^n(n+3)Q_{n+2} - (-1)^n(n-1)Q_{n+1} + 2(-1)^n(n+6)Q_n + 11)$.

(b) $\sum_{k=0}^{n}(-1)^k Q_{2k} = \frac{1}{10}((-1)^n(3Q_{2n+2} - 2Q_{2n+1} + 3Q_{2n} - 2Q_{2n-1} - 7Q_{2n-2} - 2Q_{2n-3}) + 24)$.

(c) $\sum_{k=0}^{n}(-1)^k Q_{2k+1} = \frac{1}{10}((-1)^n(Q_{2n+2} + 6Q_{2n+1} + Q_{2n} - 4Q_{2n-1} + Q_{2n-2} + 6Q_{2n-3}) - 2)$.

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4, S_5 = 8$ in the Theorem, we have the following corollary which presents sum formula of adjusted sixth-order Jacobsthal numbers.

Corollary 3.17. *For $n \geq 0$, adjusted sixth-order Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n}(-1)^k S_k = \frac{1}{9}(-(-1)^n(n+5)S_{n+5} + (-1)^n(2n+9)S_{n+4} - (-1)^n(n+2)S_{n+3} + 2(-1)^n(n+3)S_{n+2} - (-1)^n(n-1)S_{n+1} + 2(-1)^n(n+6)S_n + 1)$.

(b) $\sum_{k=0}^{n}(-1)^k S_{2k} = \frac{1}{10}((-1)^n(3S_{2n+2} - 2S_{2n+1} + 3S_{2n} - 2S_{2n-1} - 7S_{2n-2} - 2S_{2n-3}) - 1)$.

(c) $\sum_{k=0}^{n}(-1)^k S_{2k+1} = \frac{1}{10}((-1)^n(S_{2n+2} + 6S_{2n+1} + S_{2n} - 4S_{2n-1} + S_{2n-2} + 6S_{2n-3}) + 3)$.

From the last Theorem, we have the following corollary which gives sum formula of modified sixth-order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 6, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15, R_5 = 31$).

Corollary 3.18. *For $n \geq 0$, modified sixth-order Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=0}^{n}(-1)^k R_k = \frac{1}{9}(-(-1)^n(n+5)R_{n+5} + (-1)^n(2n+9)R_{n+4} - (-1)^n(n+2)R_{n+3} + 2(-1)^n(n+3)R_{n+2} - (-1)^n(n-1)R_{n+1} + 2(-1)^n(n+6)R_n - 3)$.

(b) $\sum_{k=0}^{n}(-1)^k R_{2k} = \frac{1}{10}((-1)^n(3R_{2n+2} - 2R_{2n+1} + 3R_{2n} - 2R_{2n-1} - 7R_{2n-2} - 2R_{2n-3}) + 27)$.

(c) $\sum_{k=0}^{n}(-1)^k R_{2k+1} = \frac{1}{10}((-1)^n(R_{2n+2} + 6R_{2n+1} + R_{2n} - 4R_{2n-1} + R_{2n-2} + 6R_{2n-3}) - 1)$.

Taking $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.3. *If $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k = \frac{1}{4}((-1)^n(2W_{n+1} + 5W_{n+2} + 3W_{n+4} - W_{n+5} + 13W_n) + W_5 - 3W_4 - 5W_2 - 2W_1 - 9W_0)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{2k} = \frac{1}{82}((-1)^n(5W_{2n+2} - 6W_{2n+1} + 54W_{2n} - 31W_{2n-1} - 109W_{2n-2} - 52W_{2n-3}) - 4W_5 + 13W_4 + 6W_3 - 8W_2 - 3W_1 + 17W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{2k+1} = \frac{1}{82}((-1)^n (4W_{2n+2} + 69W_{2n+1} - 6W_{2n} - 74W_{2n-1} + 3W_{2n-2} + 65W_{2n-3}) + 5W_5 - 6W_4 - 28W_3 - 31W_2 - 27W_1 - 52W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of 6-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 2$).

Corollary 3.19. *For $n \geq 0$, 6-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k G_k = \frac{1}{4}((-1)^n (2G_{n+1} + 5G_{n+2} + 3G_{n+4} - G_{n+5} + 13G_n) - 1)$.

(b) $\sum_{k=0}^{n}(-1)^k G_{2k} = \frac{1}{82}((-1)^n (5G_{2n+2} - 6G_{2n+1} + 54G_{2n} - 31G_{2n-1} - 109G_{2n-2} - 52G_{2n-3}) + 5)$.

(c) $\sum_{k=0}^{n}(-1)^k G_{2k+1} = \frac{1}{82}((-1)^n (4G_{2n+2} + 69G_{2n+1} - 6G_{2n} - 74G_{2n-1} + 3G_{2n-2} + 65G_{2n-3}) + 4)$.

Taking $W_n = H_n$ with $H_0 = 6, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542$ in the last proposition, we have the following corollary which presents sum formulas of Lucas 6-primes numbers.

Corollary 3.20. *For $n \geq 0$, Lucas 6-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k H_k = \frac{1}{4}((-1)^n (2H_{n+1} + 5H_{n+2} + 3H_{n+4} - H_{n+5} + 13H_n) - 16)$.

(b) $\sum_{k=0}^{n}(-1)^k H_{2k} = \frac{1}{82}((-1)^n (5H_{2n+2} - 6H_{2n+1} + 54H_{2n} - 31H_{2n-1} - 109H_{2n-2} - 52H_{2n-3}) + 44)$.

(c) $\sum_{k=0}^{n}(-1)^k H_{2k+1} = \frac{1}{82}((-1)^n (4H_{2n+2} + 69H_{2n+1} - 6H_{2n} - 74H_{2n-1} + 3H_{2n-2} + 65H_{2n-3}) - 14)$.

From the last proposition, we have the following corollary which gives sum formulas of modified 6-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1$).

Corollary 3.21. *For $n \geq 0$, modified 6-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k E_k = \frac{1}{4}((-1)^n (2E_{n+1} + 5E_{n+2} + 3E_{n+4} - E_{n+5} + 13E_n) - 2)$.

(b) $\sum_{k=0}^{n}(-1)^k E_{2k} = \frac{1}{82}((-1)^n (5E_{2n+2} - 6E_{2n+1} + 54E_{2n} - 31E_{2n-1} - 109E_{2n-2} - 52E_{2n-3}) + 9)$.

(c) $\sum_{k=0}^{n}(-1)^k E_{2k+1} = \frac{1}{82}((-1)^n (4E_{2n+2} + 69E_{2n+1} - 6E_{2n} - 74E_{2n-1} + 3E_{2n-2} + 65E_{2n-3}) - 1)$.

3.3 The case $x = i$

In this subsection we consider the special case $x = i$.

Taking $x = i, r = s = t = u = v = y = 1$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.4. *If $r = s = t = u = v = y = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} i^k W_k = \frac{1}{-2+i}(i^n(iW_{n+5} + (1-i)W_{n+4} - (1+2i)W_{n+3} - 2W_{n+2} + iW_{n+1} - W_n) - iW_5 - (1-i)W_4 + (1+2i)W_3 + 2W_2 - iW_1 - (1-i)W_0)$.

(b) $\sum_{k=0}^{n} i^k W_{2k} = \frac{1}{4+i}(i^n(-2iW_{2n+2} + (1+2i)W_{2n+1} + (1+3i)W_{2n} + (1+i)W_{2n-1} + (2+i)W_{2n-2} + iW_{2n-3}) + W_5 - 3W_4 + (1-i)W_3 + (1+3i)W_2 - iW_1 + (4-i)W_0)$.

(c) $\sum_{k=0}^{n} i^k W_{2k+1} = \frac{1}{4+i}(i^n(W_{2n+2} + (1+i)W_{2n+1} + (1-i)W_{2n} + (2-i)W_{2n-1} - iW_{2n-2} - 2iW_{2n-3}) - 2W_5 + (2-i)W_4 + (2+3i)W_3 + (1-i)W_2 + (5-i)W_1 + W_0)$.

From the above Proposition, we have the following Corollary which gives sum formulas of Hexanacci numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$).

Corollary 3.22. *For $n \geq 0$, Hexanacci numbers have the following properties:*

(a) $\sum_{k=0}^{n} i^k H_k = \frac{1}{-2+i}(i^n(iH_{n+5} + (1-i)H_{n+4} - (1+2i)H_{n+3} - 2H_{n+2} + iH_{n+1} - H_n) - i)$.

(b) $\sum_{k=0}^{n} i^k H_{2k} = \frac{1}{4+i}(i^n(-2iH_{2n+2} + (1+2i)H_{2n+1} + (1+3i)H_{2n} + (1+i)H_{2n-1} + (2+i)H_{2n-2} + iH_{2n-3}) - 1)$.

(c) $\sum_{k=0}^{n} i^k H_{2k+1} = \frac{1}{4+i}(i^n(H_{2n+2} + (1+i)H_{2n+1} + (1-i)H_{2n} + (2-i)H_{2n-1} - iH_{2n-2} - 2iH_{2n-3}) + 2)$.

Taking $H_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$ in the above Proposition, we have the following Corollary which presents sum formulas of Hexanacci-Lucas numbers.

Corollary 3.23. *For $n \geq 0$, Hexanacci-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} i^k E_k = \frac{1}{-2+i}(i^n(iE_{n+5} + (1-i)E_{n+4} - (1+2i)E_{n+3} - 2E_{n+2} + iE_{n+1} - E_n) + (-8+3i))$.

(b) $\sum_{k=0}^{n} i^k E_{2k} = \frac{1}{4+i}(i^n(-2iE_{2n+2} + (1+2i)E_{2n+1} + (1+3i)E_{2n} + (1+i)E_{2n-1} + (2+i)E_{2n-2} + iE_{2n-3}) + (20-5i))$.

(c) $\sum_{k=0}^{n} i^k E_{2k+1} = \frac{1}{4+i}(i^n(E_{2n+2} + (1+i)E_{2n+1} + (1-i)E_{2n} + (2-i)E_{2n-1} - iE_{2n-2} - 2iE_{2n-3}) + (-4+2i))$.

Corresponding sums of the other sixth order generalized Hexanacci numbers can be calculated similarly.

4 Sum Formulas of Generalized Hexanacci Numbers with Negative Subscripts

The following Theorem presents some summing formulas of generalized Hexanacci numbers with negative subscripts.

Theorem 4.1. *Let x be a real (or complex) number. For $n \geq 1$ we have the following formulas:*

(a) *If $y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6 \neq 0$, then*

$$\sum_{k=1}^{n} x^k W_{-k} = \frac{\Theta_4(x)}{y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6}$$

where
$\Theta_4(x) = -x^{n+1}W_{-n+5} + (r-x)x^{n+1}W_{-n+4} + (s+rx-x^2)x^{n+1}W_{-n+3} + (t+rx^2+sx-x^3)x^{n+1}W_{-n+2} + (u+rx^3+sx^2+tx-x^4)x^{n+1}W_{-n+1} + (v+rx^4+sx^3+tx^2+ux-x^5)x^{n+1}W_{-n} + xW_5 - x(r-x)W_4 + x(-s-rx+x^2)W_3 + x(-t-rx^2-sx+x^3)W_2 + x(-u-rx^3-sx^2-tx+x^4)W_1 + x(-v-rx^4-sx^3-tx^2-ux+x^5)W_0$.

(b) *If $-2sx^5 - 2ux^4 - v^2x - 2x^3y - r^2x^5 + s^2x^4 - t^2x^3 + u^2x^2 + y^2 + x^6 - 2rtx^4 - 2rvx^3 + 2sux^3 - 2tvx^2 + 2sx^2y + 2uxy \neq 0$ then*

$$\sum_{k=1}^{n} x^k W_{-2k} = \frac{\Theta_5(x)}{\Delta_2}$$

where
$\Theta_5(x) = (-y - sx^2 - ux + x^3)x^{n+1}W_{-2n+4} + x^{n+1}(tx^2 + ry + vx + rsx^2 + rux)W_{-2n+3} + (-2sx^3 - ux^2 - r^2x^3 + s^2x^2 + sy - xy + x^4 - rtx^2 - rvx + sux)x^{n+1}W_{-2n+2} + (vx^2 + ty + rux^2 - svx + tux + rxy)x^{n+1}W_{-2n+1} + (-2sx^4 + u^2x - 2ux^3 - x^2y - r^2x^4 + s^2x^3 - t^2x^2 + uy + x^5 - 2rtx^3 - rvx^2 + 2sux^2 - tvx + sxy)x^{n+1}W_{-2n} + y(v + rx^2 + tx)x^{n+1}W_{-2n-1} - x(v + rx^2 + tx)W_5 + x(y + sx^2 + r^2x^2 + rv + ux - x^3 + rtx)W_4 - x(tx^2 - sv + ry + vx + rux - stx)W_3 + x(2sx^3 + t^2x +$

$ux^2+r^2x^3-s^2x^2+tv-sy+xy-x^4+2rtx^2+rvx-sux)W_2-x(vx^2-uv+ty-svx+rxy)W_1+x(2sx^4-u^2x+2ux^3+x^2y+r^2x^4-s^2x^3+t^2x^2-uy+v^2-x^5+2rtx^3+2rvx^2-2sux^2+2tvx-sxy)W_0,$

$\Delta_2=-2sx^5-2ux^4-v^2x-2x^3y-r^2x^5+s^2x^4-t^2x^3+u^2x^2+y^2+x^6-2rtx^4-2rvx^3+2sux^3-2tvx^2+2sx^2y+2uxy.$

(c) *If* $-2sx^5-2ux^4-v^2x-2x^3y-r^2x^5+s^2x^4-t^2x^3+u^2x^2+y^2+x^6-2rtx^4-2rvx^3+2sux^3-2tvx^2+2sx^2y+2uxy\neq 0$ *then*

$$\sum_{k=1}^{n}x^kW_{-2k+1}=\frac{\Theta_6(x)}{\Delta_2}$$

where

$\Theta_6(x)=(v+rx^2+tx)x^{n+2}W_{-2n+4}+(-y-sx^2-r^2x^2-rv-ux+x^3-rtx)x^{n+2}W_{-2n+3}+(tx^2-sv+ry+vx+rux-stx)x^{n+2}W_{-2n+2}+(-2sx^3-t^2x-ux^2-r^2x^3+s^2x^2-tv+sy-xy+x^4-2rtx^2-rvx+sux)x^{n+2}W_{-2n+1}+(vx^2-uv+ty-svx+rxy)x^{n+2}W_{-2n}+y(-y-sx^2-ux+x^3)x^{n+1}W_{-2n-1}+x(y+sx^2+ux-x^3)W_5-x(tx^2+ry+vx+rsx^2+rux)W_4+x(2sx^3+ux^2+r^2x^3-s^2x^2-sy+xy-x^4+rtx^2+rvx-sux)W_3-x(vx^2+ty+rux^2-svx+tux+rxy)W_2+x(2sx^4-u^2x+2ux^3+x^2y+r^2x^4-s^2x^3+t^2x^2-uy-x^5+2rtx^3+rvx^2-2sux^2+tvx-sxy)W_1-xy(v+rx^2+tx)W_0.$

Proof.

(a) Using the recurrence relation

$$W_{-n}=\frac{1}{y}W_{-n+6}-\frac{v}{y}W_{-n+1}-\frac{u}{y}W_{-n+2}-\frac{t}{y}W_{-n+3}-\frac{s}{y}W_{-n+4}-\frac{r}{y}W_{-n+5}$$

i.e.

$$yW_{-n}=W_{-n+6}-rW_{-n+5}-sW_{-n+4}-tW_{-n+3}-uW_{-n+2}-vW_{-n+1}$$

we obtain

$$\begin{aligned}
yx^nW_{-n} &= x^nW_{-n+6}-rx^nW_{-n+5}-sx^nW_{-n+4}-tx^nW_{-n+3}-ux^nW_{-n+2}-vx^nW_{-n+1}\\
yx^{n-1}W_{-n+1} &= x^{n-1}W_{-n+7}-rx^{n-1}W_{-n+6}-sx^{n-1}W_{-n+5}-tx^{n-1}W_{-n+4}\\
&\quad -ux^{n-1}W_{-n+3}-vx^{n-1}W_{-n+2}\\
yx^{n-2}W_{-n+2} &= x^{n-2}W_{-n+8}-rx^{n-2}W_{-n+7}-sx^{n-2}W_{-n+6}-tx^{n-2}W_{-n+5}\\
&\quad -ux^{n-2}W_{-n+4}-vx^{n-2}W_{-n+3}\\
&\vdots\\
yx^3W_{-3} &= x^3W_3-rx^3W_2-sx^3W_1-tx^3W_0-ux^3W_{-1}-vx^3W_{-2}\\
yx^2W_{-2} &= x^2W_4-rx^2W_3-sx^2W_2-tx^2W_1-ux^2W_0-vx^2W_{-1}\\
yx^1W_{-1} &= x^1W_5-rx^1W_4-sx^1W_3-tx^1W_2-ux^1W_1-vx^1W_0.
\end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned} y(\sum_{k=1}^{n} x^k W_{-k}) &= (-x^{n+1}W_{-n+5} - x^{n+2}W_{-n+4} - x^{n+3}W_{-n+3} - x^{n+4}W_{-n+2} - x^{n+5}W_{-n+1} \\ &\quad -x^{n+6}W_{-n} + x^1W_5 + x^2W_4 + x^3W_3 + x^4W_2 + x^5W_1 + x^6W_0 + \sum_{k=1}^{n} x^{k+6}W_{-k}) \\ &\quad -r(-x^{n+1}W_{-n+4} - x^{n+2}W_{-n+3} - x^{n+3}W_{-n+2} - x^{n+4}W_{-n+1} - x^{n+5}W_{-n} \\ &\quad +x^1W_4 + x^2W_3 + x^3W_2 + x^4W_1 + x^5W_0 + \sum_{k=1}^{n} x^{k+5}W_{-k}) \\ &\quad -s(-x^{n+1}W_{-n+3} - x^{n+2}W_{-n+2} - x^{n+3}W_{-n+1} - x^{n+4}W_{-n} \\ &\quad +x^1W_3 + x^2W_2 + x^3W_1 + x^4W_0 + \sum_{k=1}^{n} x^{k+4}W_{-k}) \\ &\quad -t(-x^{n+1}W_{-n+2} - x^{n+2}W_{-n+1} - x^{n+3}W_{-n} + x^1W_2 + x^2W_1 + x^3W_0 + \sum_{k=1}^{n} x^{k+3}W_{-k}) \\ &\quad -u(-x^{n+1}W_{-n+1} - x^{n+2}W_{-n} + x^1W_1 + x^2W_0 + \sum_{k=1}^{n} x^{k+2}W_{-k}) \\ &\quad -v(-x^{n+1}W_{-n} + x^1W_0 + \sum_{k=1}^{n} x^{k+1}W_{-k}) \end{aligned}$$

and then the desired result follows.

(b) and (c) Using the recurrence relation

$$W_{-n+6} = rW_{-n+5} + sW_{-n+4} + tW_{-n+3} + uW_{-n+2} + vW_{-n+1} + yW_{-n}$$

i.e.

$$vW_{-n+1} = W_{-n+6} - rW_{-n+5} - sW_{-n+4} - tW_{-n+3} - uW_{-n+2} - yW_{-n}$$

we obtain

$$\begin{aligned} vx^nW_{-2n+1} &= x^nW_{-2n+6} - rx^nW_{-2n+5} - sx^nW_{-2n+4} - tx^nW_{-2n+3} - ux^nW_{-2n+2} - yx^nW_{-2n} \\ vx^{n-1}W_{-2n+3} &= x^{n-1}W_{-2n+8} - rx^{n-1}W_{-2n+7} - sx^{n-1}W_{-2n+6} - tx^{n-1}W_{-2n+5} \\ &\quad -ux^{n-1}W_{-2n+4} - yx^{n-1}W_{-2n+2} \\ vx^{n-2}W_{-2n+5} &= x^{n-2}W_{-2n+10} - rx^{n-2}W_{-2n+9} - sx^{n-2}W_{-2n+8} - tx^{n-2}W_{-2n+7} \\ &\quad -ux^{n-2}W_{-2n+6} - yx^{n-2}W_{-2n+4} \\ &\quad \vdots \\ vx^3W_{-5} &= x^3W_0 - rx^3W_{-1} - sx^3W_{-2} - tx^3W_{-3} - ux^3W_{-4} - yx^3W_{-6} \\ vx^2W_{-3} &= x^2W_2 - rx^2W_1 - sx^2W_0 - tx^2W_{-1} - ux^2W_{-2} - yx^2W_{-4} \\ vx^1W_{-1} &= x^1W_4 - rx^1W_3 - sx^1W_2 - tx^1W_1 - ux^1W_0 - yx^1W_{-2} \end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned} & v\sum_{k=1}^{n} x^kW_{-2k+1} \qquad (4.1) \\ = \; & (-x^{n+1}W_{-2n+4} - x^{n+2}W_{-2n+2} - x^{n+3}W_{-2n} + x^3W_0 + x^2W_2 + x^1W_4 + \sum_{k=1}^{n} x^{k+3}W_{-2k}) \\ & -r(-x^{n+1}W_{-2n+3} - x^{n+2}W_{-2n+1} + x^1W_3 + x^2W_1 + \sum_{k=1}^{n} x^{k+2}W_{-2k+1}) \\ & -s(-x^{n+1}W_{-2n+2} - x^{n+2}W_{-2n} + x^2W_0 + x^1W_2 + \sum_{k=1}^{n} x^{k+2}W_{-2k}) \\ & -t(-x^{n+1}W_{-2n+1} + x^1W_1 + \sum_{k=1}^{n} x^{k+1}W_{-2k+1}) \\ & -u(-x^{n+1}W_{-2n} + x^1W_0 + \sum_{k=1}^{n} x^{k+1}W_{-2k}) - y(\sum_{k=1}^{n} x^kW_{-2k}). \end{aligned}$$

Similarly, using the recurrence relation

$$W_{-n+6} = rW_{-n+5} + sW_{-n+4} + tW_{-n+3} + uW_{-n+2} + vW_{-n+1} + yW_{-n}$$

i.e.

$$vW_{-n+1} = W_{-n+6} - rW_{-n+5} - sW_{-n+4} - tW_{-n+3} - uW_{-n+2} - yW_{-n}$$

we obtain

$$\begin{aligned}
vx^nW_{-2n} &= x^nW_{-2n+5} - rx^nW_{-2n+4} - sx^nW_{-2n+3} - tx^nW_{-2n+2} - ux^nW_{-2n+1} - yx^nW_{-2n-1} \\
vx^{n-1}W_{-2n+2} &= x^{n-1}W_{-2n+7} - rx^{n-1}W_{-2n+6} - sx^{n-1}W_{-2n+5} - tx^{n-1}W_{-2n+4} \\
&\quad -ux^{n-1}W_{-2n+3} - yx^{n-1}W_{-2n+1} \\
vx^{n-2}W_{-2n+4} &= x^{n-2}W_{-2n+9} - rx^{n-2}W_{-2n+8} - sx^{n-2}W_{-2n+7} - tx^{n-2}W_{-2n+6} \\
&\quad -ux^{n-2}W_{-2n+5} - yx^{n-2}W_{-2n+3} \\
&\vdots \\
vx^3W_{-6} &= x^3W_{-1} - rx^3W_{-2} - sx^3W_{-3} - tx^3W_{-4} - ux^3W_{-5} - yx^3W_{-7} \\
vx^2W_{-4} &= x^2W_1 - rx^2W_0 - sx^2W_{-1} - tx^2W_{-2} - ux^2W_{-3} - yx^2W_{-5} \\
vx^1W_{-2} &= x^1W_3 - rx^1W_2 - sx^1W_1 - tx^1W_0 - ux^1W_{-1} - yx^1W_{-3}
\end{aligned}$$

If we add the equations side by side, we get

$$\begin{aligned}
v\sum_{k=1}^{n} x^kW_{-2k} &= (-x^{n+1}W_{-2n+3} - x^{n+2}W_{-2n+1} + x^2W_1 + x^1W_3 + \sum_{k=1}^{n} x^{k+2}W_{-2k+1}) \\
&\quad -r(-x^{n+1}W_{-2n+2} - x^{n+2}W_{-2n} + x^1W_2 + x^2W_0 + \sum_{k=1}^{n} x^{k+2}W_{-2k}) \\
&\quad -s(-x^{n+1}W_{-2n+1} + x^1W_1 + \sum_{k=1}^{n} x^{k+1}W_{-2k+1}) \\
&\quad -t(-x^{n+1}W_{-2n} + x^1W_0 + \sum_{k=1}^{n} x^{k+1}W_{-2k}) - u(\sum_{k=1}^{n} x^kW_{-2k+1}) \\
&\quad -y(x^nW_{-2n-1} - x^0W_{-1} + \sum_{k=1}^{n} x^{k-1}W_{-2k+1}).
\end{aligned} \tag{4.2}$$

□

5 Specific Cases

In this section, for the specific cases of x, we present the closed form solutions (identities) of the sums $\sum_{k=1}^{n} x^kW_{-k}$, $\sum_{k=1}^{n} x^kW_{-2k}$ and $\sum_{k=1}^{n} x^kW_{-2k+1}$ for the specific case of sequence $\{W_n\}$.

5.1 The case $x = 1$

In this subsection we consider the special case $x = 1$.

The case $x = 1$ of Theorem 2.1 is given in Soykan [31, Theorem 3.1]. For the generalized 6-primes sequence case ($x = 1, r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$), see [7].

We only consider the case $x = 1, r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ (which is not considered in [31]).

Observe that setting $x = 1, r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ (i.e. for the generalized sixth order Jacobsthal sequence case) in Theorem 4.1 (b), (c) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

Theorem 5.1. *If $r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = \frac{1}{6}(-W_{-n+5} + W_{-n+3} + 2W_{-n+2} + 3W_{-n+1} + 4W_{-n} + W_5 - W_3 - 2W_2 - 3W_1 - 4W_0)$.

(b) $\sum_{k=1}^{n} W_{-2k} = \frac{1}{9}((n+1)W_{-2n+4} - 2(n+2)W_{-2n+3} + (n+4)W_{-2n+2} - 2(n+2)W_{-2n+1} + (7+n)W_{-2n} - 2(n+2)W_{-2n-1} + 2W_5 - 3W_4 + 2W_3 - 6W_2 + 2W_1 - 9W_0)$.

(c) $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{9}(-(n+3)W_{-2n+4} + (2n+5)W_{-2n+3} - (n+3)W_{-2n+2} + 2(n+4)W_{-2n+1} - (n+3)W_{-2n} + 2(n+1)W_{-2n-1} - W_5 + 4W_4 - 4W_3 + 4W_2 - 7W_1 + 4W_0)$.

Proof.

(a) We use Theorem 4.1 (a). If we set $x = 1, r = 1, s = 1, t = 1, u = 2$ in Theorem 4.1 (a) we get (a).

(b) We use Theorem 4.1 (b). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 4.1 (b) then we have

$$\sum_{k=1}^{n} x^k W_{-2k} = \frac{g_4(x)}{(x-1)(x-4)(x+x^2+1)^2}$$

where
$g_4(x) = -x(x+x^2+1)(-x^n(x-2)W_{-2n+4} - 2x^nW_{-2n+3} - x^n(-4x+x^2+2)W_{-2n+2} - 2x^nW_{-2n+1} - x^n(-4x^2+x^3+2)W_{-2n} - 2x^nW_{-2n-1} + W_5 + (x-3)W_4 + W_3 + (x^2+1-4x)W_2 + W_1 + (1-4x^2+x^3)W_0)$.
For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

$$\begin{aligned}\sum_{k=1}^{n} W_{-2k} &= \left.\frac{\frac{d}{dx}(g_4(x))}{\frac{d}{dx}\left((x-1)(x-4)(x+x^2+1)^2\right)}\right|_{x=1} \\ &= \frac{1}{9}((n+1)W_{-2n+4} - 2(n+2)W_{-2n+3} + (n+4)W_{-2n+2} - 2(n+2)W_{-2n+1} \\ &\quad +(7+n)W_{-2n} - 2(n+2)W_{-2n-1} + 2W_5 - 3W_4 + 2W_3 - 6W_2 + 2W_1 - 9W_0).\end{aligned}$$

(c) We use Theorem 4.1 (c). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 4.1 (c) then we have

$$\sum_{k=1}^{n} x^k W_{-2k+1} = \frac{g_5(x)}{(x-1)(x-4)(x+x^2+1)^2}$$

where
$g_5(x) = -x(x+x^2+1)(-x^{n+1}W_{-2n+4} - x^{n+1}(x-3)W_{-2n+3} - x^{n+1}W_{-2n+2} - x^{n+1}(-4x+x^2+1)W_{-2n+1} - x^{n+1}W_{-2n} - 2x^n(x-2)W_{-2n-1} + (x-2)W_5 + 2W_4 + (x^2+2-4x)W_3 + 2W_2 + (2-4x^2+x^3)W_1 + 2W_0)$.
For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get (c) by using

$$\begin{aligned}\sum_{k=1}^{n} W_{-2k+1} &= \left.\frac{\frac{d}{dx}(g_5(x))}{\frac{d}{dx}\left((x-1)(x-4)(x+x^2+1)^2\right)}\right|_{x=1} \\ &= \frac{1}{9}(-(n+3)W_{-2n+4} + (2n+5)W_{-2n+3} - (n+3)W_{-2n+2} + 2(n+4)W_{-2n+1} \\ &\quad -(n+3)W_{-2n} + 2(n+1)W_{-2n-1} - W_5 + 4W_4 - 4W_3 + 4W_2 - 7W_1 + 4W_0).\end{aligned}$$

□

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1, J_5 = 1$ in the last Proposition, we have the following Corollary which presents sum formulas of sixth-order Jacobsthal numbers.

Corollary 5.2. *For $n \geq 1$, sixth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=1}^{n} J_{-k} = \frac{1}{6}(-J_{-n+5} + J_{-n+3} + 2J_{-n+2} + 3J_{-n+1} + 4J_{-n} - 5)$.

(b) $\sum_{k=1}^{n} J_{-2k} = \frac{1}{9}((n+1)J_{-2n+4} - 2(n+2)J_{-2n+3} + (n+4)J_{-2n+2} - 2(n+2)J_{-2n+1} + (7+n)J_{-2n} - 2(n+2)J_{-2n-1} - 3)$.

(c) $\sum_{k=1}^{n} J_{-2k+1} = \frac{1}{9}(-(n+3)J_{-2n+4} + (2n+5)J_{-2n+3} - (n+3)J_{-2n+2} + 2(n+4)J_{-2n+1} - (n+3)J_{-2n} + 2(n+1)J_{-2n-1} - 4)$.

From the last Theorem, we have the following Corollary which gives sum formulas of sixth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20, j_5 = 40$).

Corollary 5.3. *For $n \geq 1$, sixth order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} j_{-k} = \frac{1}{6}(-j_{-n+5} + j_{-n+3} + 2j_{-n+2} + 3j_{-n+1} + 4j_{-n} + 9)$.

(b) $\sum_{k=1}^{n} j_{-2k} = \frac{1}{9}((n+1)j_{-2n+4} - 2(n+2)j_{-2n+3} + (n+4)j_{-2n+2} - 2(n+2)j_{-2n+1} + (7+n)j_{-2n} - 2(n+2)j_{-2n-1} - 6)$.

(c) $\sum_{k=1}^{n} j_{-2k+1} = \frac{1}{9}(-(n+3)j_{-2n+4} + (2n+5)j_{-2n+3} - (n+3)j_{-2n+2} + 2(n+4)j_{-2n+1} - (n+3)j_{-2n} + 2(n+1)j_{-2n-1} + 21)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20, K_5 = 40$ in the last Theorem, we have the following corollary which presents sum formula of modified sixth order Jacobsthal numbers.

Corollary 5.4. *For $n \geq 1$,modified sixth order Jacobsthal numbers have the following property:*

(a) $\sum_{k=1}^{n} K_{-k} = \frac{1}{6}(-K_{-n+5} + K_{-n+3} + 2K_{-n+2} + 3K_{-n+1} + 4K_{-n} + 9)$.

(b) $\sum_{k=1}^{n} K_{-2k} = \frac{1}{9}((n+1)K_{-2n+4} - 2(n+2)K_{-2n+3} + (n+4)K_{-2n+2} - 2(n+2)K_{-2n+1} + (7+n)K_{-2n} - 2(n+2)K_{-2n-1} - 3)$.

(c) $\sum_{k=1}^{n} K_{-2k+1} = \frac{1}{9}(-(n+3)K_{-2n+4} + (2n+5)K_{-2n+3} - (n+3)K_{-2n+2} + 2(n+4)K_{-2n+1} - (n+3)K_{-2n} + 2(n+1)K_{-2n-1} + 17)$.

From the last Theorem, we have the following corollary which gives sum formula of sixth-order Jacobsthal Perrin numbers (take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16, Q_5 = 32$).

Corollary 5.5. *For $n \geq 1$, sixth-order Jacobsthal Perrin numbers have the following property:*

(a) $\sum_{k=1}^{n} Q_{-k} = \frac{1}{6}(-Q_{-n+5} + Q_{-n+3} + 2Q_{-n+2} + 3Q_{-n+1} + 4Q_{-n} + 8)$.

(b) $\sum_{k=1}^{n} Q_{-2k} = \frac{1}{9}((n+1)Q_{-2n+4} - 2(n+2)Q_{-2n+3} + (n+4)Q_{-2n+2} - 2(n+2)Q_{-2n+1} + (7+n)Q_{-2n} - 2(n+2)Q_{-2n-1} - 7)$.

(c) $\sum_{k=1}^{n} Q_{-2k+1} = \frac{1}{9}(-(n+3)Q_{-2n+4} + (2n+5)Q_{-2n+3} - (n+3)Q_{-2n+2} + 2(n+4)Q_{-2n+1} - (n+3)Q_{-2n} + 2(n+1)Q_{-2n-1} + 20)$.

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4, S_5 = 8$ in the last Theorem, we have the following corollary which presents sum formula of adjusted sixth-order Jacobsthal numbers.

Corollary 5.6. *For $n \geq 1$, adjusted sixth-order Jacobsthal numbers have the following property:*

(a) $\sum_{k=1}^{n} S_{-k} = \frac{1}{6}(-S_{-n+5} + S_{-n+3} + 2S_{-n+2} + 3S_{-n+1} + 4S_{-n} + 1)$.

(b) $\sum_{k=1}^{n} S_{-2k} = \frac{1}{9}((n+1)S_{-2n+4} - 2(n+2)S_{-2n+3} + (n+4)S_{-2n+2} - 2(n+2)S_{-2n+1} + (7+n)S_{-2n} - 2(n+2)S_{-2n-1} + 4)$.

(c) $\sum_{k=1}^{n} S_{-2k+1} = \frac{1}{9}(-(n+3)S_{-2n+4} + (2n+5)S_{-2n+3} - (n+3)S_{-2n+2} + 2(n+4)S_{-2n+1} - (n+3)S_{-2n} + 2(n+1)S_{-2n-1} - 3)$.

From the last Theorem, we have the following corollary which gives sum formula of modified sixth-order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 6, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15, R_5 = 31$).

Corollary 5.7. *For $n \geq 1$, modified sixth-order Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=1}^{n} R_{-k} = \frac{1}{6}(-R_{-n+5} + R_{-n+3} + 2R_{-n+2} + 3R_{-n+1} + 4R_{-n} - 9)$.

(b) $\sum_{k=1}^{n} R_{-2k} = \frac{1}{9}((n+1)R_{-2n+4} - 2(n+2)R_{-2n+3} + (n+4)R_{-2n+2} - 2(n+2)R_{-2n+1} + (7+n)R_{-2n} - 2(n+2)R_{-2n-1} - 39)$.

(c) $\sum_{k=1}^{n} R_{-2k+1} = \frac{1}{9}(-(n+3)R_{-2n+4} + (2n+5)R_{-2n+3} - (n+3)R_{-2n+2} + 2(n+4)R_{-2n+1} - (n+3)R_{-2n} + 2(n+1)R_{-2n-1} + 30)$.

5.2 The case $x = -1$

In this subsection we consider the special case $x = -1$.

Taking $r = s = t = u = v = y = 1$ in Theorem 4.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 5.1. *If $r = s = t = u = v = y = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} (-1)^k W_{-k} = (-1)^n (-W_{-n+5} + 2W_{-n+4} - W_{-n+3} + 2W_{-n+2} - W_{-n+1} + 2W_{-n}) + W_5 - 2W_4 + W_3 - 2W_2 + W_1 - 2W_0$.

(b) $\sum_{k=1}^{n} (-1)^k W_{-2k} = \frac{1}{5}((-1)^n (2W_{-2n+4} - W_{-2n+3} - 5W_{-2n+2} - 2W_{-2n+1} + 2W_{-2n} - W_{-2n-1}) + W_5 - 3W_4 + 4W_2 + W_1 - 3W_0)$.

(c) $\sum_{k=1}^{n} (-1)^k W_{-2k+1} = \frac{1}{5}((-1)^n (W_{-2n+4} - 3W_{-2n+3} + 4W_{-2n+1} + W_{-2n} + 2W_{-2n-1}) - 2W_5 + W_4 + 5W_3 + 2W_2 - 2W_1 + W_0)$.

From the above Proposition, we have the following Corollary which gives sum formulas of Hexanacci numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$).

Corollary 5.8. *For $n \geq 1$, Hexanacci numbers have the following properties:*

(a) $\sum_{k=1}^{n} (-1)^k H_{-k} = (-1)^n (-H_{-n+5} + 2H_{-n+4} - H_{-n+3} + 2H_{-n+2} - H_{-n+1} + 2H_{-n}) + H_5 - 2H_4 + H_3 - 2H_2 + H_1 - 2H_0$.

(b) $\sum_{k=1}^{n} (-1)^k H_{-2k} = \frac{1}{5}((-1)^n (2H_{-2n+4} - H_{-2n+3} - 5H_{-2n+2} - 2H_{-2n+1} + 2H_{-2n} - H_{-2n-1}) + H_5 - 3H_4 + 4H_2 + H_1 - 3H_0)$.

(c) $\sum_{k=1}^{n} (-1)^k H_{-2k+1} = \frac{1}{5}((-1)^n (H_{-2n+4} - 3H_{-2n+3} + 4H_{-2n+1} + H_{-2n} + 2H_{-2n-1}) - 2H_5 + H_4 + 5H_3 + 2H_2 - 2H_1 + H_0)$.

Taking $W_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$ in the above Proposition, we have the following Corollary which presents sum formulas of Hexanacci-Lucas numbers.

Corollary 5.9. *For $n \geq 1$, Hexanacci-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} (-1)^k E_{-k} = (-1)^n (-E_{-n+5} + 2E_{-n+4} - E_{-n+3} + 2E_{-n+2} - E_{-n+1} + 2E_{-n}) + E_5 - 2E_4 + E_3 - 2E_2 + E_1 - 2E_0$.

(b) $\sum_{k=1}^{n} (-1)^k E_{-2k} = \frac{1}{5}((-1)^n (2E_{-2n+4} - E_{-2n+3} - 5E_{-2n+2} - 2E_{-2n+1} + 2E_{-2n} - E_{-2n-1}) + E_5 - 3E_4 + 4E_2 + E_1 - 3E_0)$.

(c) $\sum_{k=1}^{n} (-1)^k E_{-2k+1} = \frac{1}{5}((-1)^n (E_{-2n+4} - 3E_{-2n+3} + 4E_{-2n+1} + E_{-2n} + 2E_{-2n-1}) - 2E_5 + E_4 + 5E_3 + 2E_2 - 2E_1 + E_0)$.

Taking $r = 2, s = t = u = v = y = 1$ in Theorem 4.1 (a), (b) and (c), we obtain the following Proposition.

Proposition 5.2. *If $r = 2, s = t = u = v = y = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}(-1)^k W_{-k} = \frac{1}{2}((-1)^n(-W_{-n+5} + 3W_{-n+4} - 2W_{-n+3} + 3W_{-n+2} - 2W_{-n+1} + 3W_{-n}) + W_5 - 3W_4 + 2W_3 - 3W_2 + 2W_1 - 3W_0)$.

(b) $\sum_{k=1}^{n}(-1)^k W_{-2k} = \frac{1}{4}((-1)^n(W_{-2n+4} - W_{-2n+3} - 4W_{-2n+2} - W_{-2n+1} + 2W_{-2n} - W_{-2n-1}) + W_5 - 3W_4 + 3W_2 - 3W_0)$.

(c) $\sum_{k=1}^{n}(-1)^k W_{-2k+1} = \frac{1}{4}((-1)^n(W_{-2n+4} - 3W_{-2n+3} + 3W_{-2n+1} + W_{-2n-1}) - W_5 + W_4 + 4W_3 + W_2 - 2W_1 + W_0)$.

From the last Proposition, we have the following Corollary which gives sum formulas of sixth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13, P_5 = 34$).

Corollary 5.10. *For $n \geq 1$, sixth-order Pell numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^k P_{-k} = \frac{1}{2}((-1)^n(-P_{-n+5} + 3P_{-n+4} - 2P_{-n+3} + 3P_{-n+2} - 2P_{-n+1} + 3P_{-n}) + 1)$.

(b) $\sum_{k=1}^{n}(-1)^k P_{-2k} = \frac{1}{4}((-1)^n(P_{-2n+4} - P_{-2n+3} - 4P_{-2n+2} - P_{-2n+1} + 2P_{-2n} - P_{-2n-1}) + 1)$.

(c) $\sum_{k=1}^{n}(-1)^k P_{-2k+1} = \frac{1}{4}((-1)^n(P_{-2n+4} - 3P_{-2n+3} + 3P_{-2n+1} + P_{-2n-1}) - 1)$.

Taking $W_n = Q_n$ with $Q_0 = 6, Q_1 = 2, Q_2 = 6, Q_3 = 17, Q_4 = 46, Q_5 = 122$ in the last Proposition, we have the following Corollary which presents sum formulas of sixth-order Pell-Lucas numbers.

Corollary 5.11. *For $n \geq 1$, sixth-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^k Q_{-k} = \frac{1}{2}((-1)^n(-Q_{-n+5} + 3Q_{-n+4} - 2Q_{-n+3} + 3Q_{-n+2} - 2Q_{-n+1} + 3Q_{-n}) - 14)$.

(b) $\sum_{k=1}^{n}(-1)^k Q_{-2k} = \frac{1}{4}((-1)^n(Q_{-2n+4} - Q_{-2n+3} - 4Q_{-2n+2} - Q_{-2n+1} + 2Q_{-2n} - Q_{-2n-1}) - 16)$.

(c) $\sum_{k=1}^{n}(-1)^k Q_{-2k+1} = \frac{1}{4}((-1)^n(Q_{-2n+4} - 3Q_{-2n+3} + 3Q_{-2n+1} + Q_{-2n-1}))$.

Observe that setting $x = -1, r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ (i.e. for the generalized sixth order Jacobsthal case) in Theorem 4.1 (a), makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

Theorem 5.12. *If $r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}(-1)^k W_{-k} = \frac{1}{9}(-(-1)^n(n+1)W_{-n+5} + (-1)^n(2n+3)W_{-n+4} - (-1)^n(n+4)W_{-n+3} + 2(-1)^n(n+3)W_{-n+2} - (-1)^n(n+7)W_{-n+1} + (-1)^n(2n+9)W_{-n} + W_5 - 3W_4 + 4W_3 - 6W_2 + 7W_1 - 9W_0)$.

(b) $\sum_{k=1}^{n}(-1)^k W_{-2k} = \frac{1}{10}((-1)^n(3W_{-2n+4} - 2W_{-2n+3} - 7W_{-2n+2} - 2W_{-2n+1} + 3W_{-2n} - 2W_{-2n-1}) + W_5 - 4W_4 + W_3 + 6W_2 + W_1 - 4W_0)$.

(c) $\sum_{k=1}^{n}(-1)^k W_{-2k+1} = \frac{1}{10}((-1)^n(W_{-2n+4} - 4W_{-2n+3} + W_{-2n+2} + 6W_{-2n+1} + W_{-2n} + 6W_{-2n-1}) - 3W_5 + 2W_4 + 7W_3 + 2W_2 - 3W_1 + 2W_0)$.

Proof.

(a) We use Theorem 4.1 (a). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 4.1 (a) then we have

$$\sum_{k=1}^{n} x^k W_{-k} = \frac{g_6(x)}{-(x-2)(x+1)(x+x^2+1)(-x+x^2+1)}$$

where

$g_6(x) = -x^{n+1}W_{-n+5} - x^{n+1}(x-1)W_{-n+4} + x^{n+1}(-x^2+x+1)W_{-n+3} + x^{n+1}(-x^3+x^2+x+1)W_{-n+2} + x^{n+1}(-x^4+x^3+x^2+x+1)W_{-n+1} + x^{n+1}(-x^5+x^4+x^3+x^2+x+1)W_{-n} + xW_5 + x(x-1)W_4 - x(-x^2+x+1)W_3 - x(-x^3+x^2+x+1)W_2 - x(-x^4+x^3+x^2+x+1)W_1 - x(-x^5+x^4+x^3+x^2+x+1)W_0.$

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) using

$$\begin{aligned}
\sum_{k=1}^{n}(-1)^k W_{-k} &= \left.\frac{\frac{d}{dx}\left(g_6(x)\right)}{\frac{d}{dx}\left(-(x-2)(x+1)(x+x^2+1)(-x+x^2+1)\right)}\right|_{x=-1} \\
&= \frac{1}{9}(-(-1)^n(n+1)W_{-n+5} + (-1)^n(2n+3)W_{-n+4} \\
&\quad - (-1)^n(n+4)W_{-n+3} + 2(-1)^n(n+3)W_{-n+2} - (-1)^n(n+7)W_{-n+1} \\
&\quad + (-1)^n(2n+9)W_{-n} + W_5 - 3W_4 + 4W_3 - 6W_2 + 7W_1 - 9W_0).
\end{aligned}$$

(b) Take $x = -1, r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$ in Theorem 4.1 (b).

(c) Take $x = -1, r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$ in Theorem 4.1 (b). □

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1, J_5 = 1$ in the last Proposition, we have the following Corollary which presents sum formulas of sixth-order Jacobsthal numbers.

Corollary 5.13. *For $n \geq 1$, sixth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^k J_{-k} = \frac{1}{9}(-(-1)^n(n+1)J_{-n+5} + (-1)^n(2n+3)J_{-n+4} - (-1)^n(n+4)J_{-n+3} + 2(-1)^n(n+3)J_{-n+2} - (-1)^n(n+7)J_{-n+1} + (-1)^n(2n+9)J_{-n} + 3).$

(b) $\sum_{k=1}^{n}(-1)^k J_{-2k} = \frac{1}{10}((-1)^n(3J_{-2n+4} - 2J_{-2n+3} - 7J_{-2n+2} - 2J_{-2n+1} + 3J_{-2n} - 2J_{-2n-1}) + 5).$

(c) $\sum_{k=1}^{n}(-1)^k J_{-2k+1} = \frac{1}{10}((-1)^n(J_{-2n+4} - 4J_{-2n+3} + J_{-2n+2} + 6J_{-2n+1} + J_{-2n} + 6J_{-2n-1}) + 5).$

From the last Theorem, we have the following Corollary which gives sum formulas of sixth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20, j_5 = 40$).

Corollary 5.14. *For $n \geq 1$, sixth order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^k j_{-k} = \frac{1}{9}(-(-1)^n(n+1)j_{-n+5} + (-1)^n(2n+3)j_{-n+4} - (-1)^n(n+4)j_{-n+3} + 2(-1)^n(n+3)j_{-n+2} - (-1)^n(n+7)j_{-n+1} + (-1)^n(2n+9)j_{-n} - 21).$

(b) $\sum_{k=1}^{n}(-1)^k j_{-2k} = \frac{1}{10}((-1)^n(3j_{-2n+4} - 2j_{-2n+3} - 7j_{-2n+2} - 2j_{-2n+1} + 3j_{-2n} - 2j_{-2n-1}) - 7).$

(c) $\sum_{k=1}^{n}(-1)^k j_{-2k+1} = \frac{1}{10}((-1)^n(j_{-2n+4} - 4j_{-2n+3} + j_{-2n+2} + 6j_{-2n+1} + j_{-2n} + 6j_{-2n-1}) + 1).$

Taking $j_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20, K_5 = 40$ in the last Theorem, we have the following corollary which presents sum formula of modified sixth order Jacobsthal numbers.

Corollary 5.15. *For $n \geq 1$,modified sixth order Jacobsthal numbers have the following property:*

(a) $\sum_{k=1}^{n}(-1)^k K_{-k} = \frac{1}{9}(-(-1)^n(n+1)K_{-n+5} + (-1)^n(2n+3)K_{-n+4} - (-1)^n(n+4)K_{-n+3} + 2(-1)^n(n+3)K_{-n+2} - (-1)^n(n+7)K_{-n+1} + (-1)^n(2n+9)K_{-n} - 18).$

(b) $\sum_{k=1}^{n}(-1)^k K_{-2k} = \frac{1}{10}((-1)^n(3K_{-2n+4} - 2K_{-2n+3} - 7K_{-2n+2} - 2K_{-2n+1} + 3K_{-2n} - 2K_{-2n-1}) - 23).$

(c) $\sum_{k=1}^{n}(-1)^k K_{-2k+1} = \frac{1}{10}((-1)^n(K_{-2n+4} - 4K_{-2n+3} + K_{-2n+2} + 6K_{-2n+1} + K_{-2n} + 6K_{-2n-1}) - 1).$

From the last Theorem, we have the following corollary which gives sum formula of sixth-order Jacobsthal Perrin numbers (take $K_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16, Q_5 = 32$).

Corollary 5.16. *For $n \geq 1$, sixth-order Jacobsthal Perrin numbers have the following property:*

(a) $\sum_{k=1}^{n}(-1)^{k}Q_{-k} = \frac{1}{9}(-(-1)^{n}(n+1)Q_{-n+5} + (-1)^{n}(2n+3)Q_{-n+4} - (-1)^{n}(n+4)Q_{-n+3} + 2(-1)^{n}(n+3)Q_{-n+2} - (-1)^{n}(n+7)Q_{-n+1} + (-1)^{n}(2n+9)Q_{-n} - 23)$.

(b) $\sum_{k=1}^{n}(-1)^{k}Q_{-2k} = \frac{1}{10}((-1)^{n}(3Q_{-2n+4} - 2Q_{-2n+3} - 7Q_{-2n+2} - 2Q_{-2n+1} + 3Q_{-2n} - 2Q_{-2n-1}) - 24)$.

(c) $\sum_{k=1}^{n}(-1)^{k}Q_{-2k+1} = \frac{1}{10}((-1)^{n}(Q_{-2n+4} - 4Q_{-2n+3} + Q_{-2n+2} + 6Q_{-2n+1} + Q_{-2n} + 6Q_{-2n-1}) + 2)$.

Taking $Q_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4, S_5 = 8$ in the Theorem, we have the following corollary which presents sum formula of adjusted sixth-order Jacobsthal numbers.

Corollary 5.17. *For $n \geq 1$, adjusted sixth-order Jacobsthal numbers have the following property:*

(a) $\sum_{k=1}^{n}(-1)^{k}S_{-k} = \frac{1}{9}(-(-1)^{n}(n+1)S_{-n+5} + (-1)^{n}(2n+3)S_{-n+4} - (-1)^{n}(n+4)S_{-n+3} + 2(-1)^{n}(n+3)S_{-n+2} - (-1)^{n}(n+7)S_{-n+1} + (-1)^{n}(2n+9)S_{-n} + 5)$.

(b) $\sum_{k=1}^{n}(-1)^{k}S_{-2k} = \frac{1}{10}((-1)^{n}(3S_{-2n+4} - 2S_{-2n+3} - 7S_{-2n+2} - 2S_{-2n+1} + 3S_{-2n} - 2S_{-2n-1}) + 1)$.

(c) $\sum_{k=1}^{n}(-1)^{k}S_{-2k+1} = \frac{1}{10}((-1)^{n}(S_{-2n+4} - 4S_{-2n+3} + S_{-2n+2} + 6S_{-2n+1} + S_{-2n} + 6S_{-2n-1}) - 3)$.

From the last Theorem, we have the following corollary which gives sum formula of modified sixth-order Jacobsthal-Lucas numbers (take $S_n = R_n$ with $R_0 = 6, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15, R_5 = 31$).

Corollary 5.18. *For $n \geq 1$, modified sixth-order Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=1}^{n}(-1)^{k}R_{-k} = \frac{1}{9}(-(-1)^{n}(n+1)R_{-n+5} + (-1)^{n}(2n+3)R_{-n+4} - (-1)^{n}(n+4)R_{-n+3} + 2(-1)^{n}(n+3)R_{-n+2} - (-1)^{n}(n+7)R_{-n+1} + (-1)^{n}(2n+9)R_{-n} - 51)$.

(b) $\sum_{k=1}^{n}(-1)^{k}R_{-2k} = \frac{1}{10}((-1)^{n}(3R_{-2n+4} - 2R_{-2n+3} - 7R_{-2n+2} - 2R_{-2n+1} + 3R_{-2n} - 2R_{-2n-1}) - 27)$.

(c) $\sum_{k=1}^{n}(-1)^{k}R_{-2k+1} = \frac{1}{10}((-1)^{n}(R_{-2n+4} - 4R_{-2n+3} + R_{-2n+2} + 6R_{-2n+1} + R_{-2n} + 6R_{-2n-1}) + 1)$.

Taking $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ in Theorem 4.1 (a), (b) and (c), we obtain the following proposition.

Proposition 5.3. *If $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}(-1)^{k}W_{-k} = \frac{1}{4}((-1)^{n}(W_{-n+5} - 3W_{-n+4} - 5W_{-n+2} - 2W_{-n+1} - 9W_{-n}) - W_5 + 3W_4 + 5W_2 + 2W_1 + 9W_0)$.

(b) $\sum_{k=1}^{n}(-1)^{k}W_{-2k} = \frac{1}{82}((-1)^{n}(5W_{-2n+4} - 6W_{-2n+3} - 28W_{-2n+2} - 31W_{-2n+1} - 27W_{-2n} - 52W_{-2n-1}) + 4W_5 - 13W_4 - 6W_3 + 8W_2 + 3W_1 - 17W_0)$.

(c) $\sum_{k=1}^{n}(-1)^{k}W_{-2k+1} = \frac{1}{82}((-1)^{n}(4W_{-2n+4} - 13W_{-2n+3} - 6W_{-2n+2} + 8W_{-2n+1} + 3W_{-2n} + 65W_{-2n-1}) - 5W_5 + 6W_4 + 28W_3 + 31W_2 + 27W_1 + 52W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of 6-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 2$).

Corollary 5.19. *For $n \geq 1$, 6-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^{k}G_{-k} = \frac{1}{4}((-1)^{n}(G_{-n+5} - 3G_{-n+4} - 5G_{-n+2} - 2G_{-n+1} - 9G_{-n}) + 1)$.

(b) $\sum_{k=1}^{n}(-1)^k G_{-2k} = \frac{1}{82}((-1)^n (5G_{-2n+4} - 6G_{-2n+3} - 28G_{-2n+2} - 31G_{-2n+1} - 27G_{-2n} - 52G_{-2n-1}) - 5)$.

(c) $\sum_{k=1}^{n}(-1)^k G_{-2k+1} = \frac{1}{82}((-1)^n (4G_{-2n+4} - 13G_{-2n+3} - 6G_{-2n+2} + 8G_{-2n+1} + 3G_{-2n} + 65G_{-2n-1}) - 4)$.

Taking $G_n = H_n$ with $H_0 = 6, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542$ in the last proposition, we have the following corollary which presents sum formulas of Lucas 6-primes numbers.

Corollary 5.20. *For $n \geq 1$, Lucas 6-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^k H_{-k} = \frac{1}{4}((-1)^n (H_{-n+5} - 3H_{-n+4} - 5H_{-n+2} - 2H_{-n+1} - 9H_{-n}) + 16)$.

(b) $\sum_{k=1}^{n}(-1)^k H_{-2k} = \frac{1}{82}((-1)^n (5H_{-2n+4} - 6H_{-2n+3} - 28H_{-2n+2} - 31H_{-2n+1} - 27H_{-2n} - 52H_{-2n-1}) - 44)$.

(c) $\sum_{k=1}^{n}(-1)^k H_{-2k+1} = \frac{1}{82}((-1)^n (4H_{-2n+4} - 13H_{-2n+3} - 6H_{-2n+2} + 8H_{-2n+1} + 3H_{-2n} + 65H_{-2n-1}) + 14)$.

From the last proposition, we have the following corollary which gives sum formulas of modified 6-primes numbers (take $H_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1$).

Corollary 5.21. *For $n \geq 1$, modified 6-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^k E_{-k} = \frac{1}{4}((-1)^n (E_{-n+5} - 3E_{-n+4} - 5E_{-n+2} - 2E_{-n+1} - 9E_{-n}) + 2)$.

(b) $\sum_{k=1}^{n}(-1)^k E_{-2k} = \frac{1}{82}((-1)^n (5E_{-2n+4} - 6E_{-2n+3} - 28E_{-2n+2} - 31E_{-2n+1} - 27E_{-2n} - 52E_{-2n-1}) - 9)$.

(c) $\sum_{k=1}^{n}(-1)^k E_{-2k+1} = \frac{1}{82}((-1)^n (4E_{-2n+4} - 13E_{-2n+3} - 6E_{-2n+2} + 8E_{-2n+1} + 3E_{-2n} + 65E_{-2n-1}) + 1)$.

5.3 The case $x = i$

In this subsection, we consider the special case $x = i$.

Taking $r = s = t = u = v = y = 1$ in Theorem 4.1, we obtain the following proposition.

Proposition 5.4. *If $r = s = t = u = v = y = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} i^k W_{-k} = \frac{1}{2+i}(i^n(-iW_{-n+5} + (1+i)W_{-n+4} - (1-2i)W_{-n+3} - 2W_{-n+2} - iW_{-n+1} + (1+i)W_{-n}) + iW_5 - (1+i)W_4 + (1-2i)W_3 + 2W_2 + iW_1 - (1+i)W_0)$.

(b) $\sum_{k=1}^{n} i^k W_{-2k} = \frac{1}{-4+i}(i^n(2W_{-2n+4} - (2+i)W_{-2n+3} - (2-3i)W_{-2n+2} - (1+i)W_{-2n+1} - (5+i)W_{-2n} - W_{-2n-1}) + W_5 - 3W_4 + (1+i)W_3 + (1-3i)W_2 + iW_1 + (4+i)W_0)$.

(c) $\sum_{k=1}^{n} i^k W_{-2k+1} = \frac{1}{-4+i}(i^n(-iW_{-2n+4} + 3iW_{-2n+3} + (1-i)W_{-2n+2} - (3+i)W_{-2n+1} + W_{-2n} + 2W_{-2n-1}) - 2W_5 + (2+i)W_4 + (2-3i)W_3 + (1+i)W_2 + (5+i)W_1 + W_0)$.

From the above Proposition, we have the following Corollary which gives sum formulas of Hexanacci numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$).

Corollary 5.22. *For $n \geq 1$, Hexanacci numbers have the following properties:*

(a) $\sum_{k=1}^{n} i^k H_{-k} = \frac{1}{2+i}(i^n(-iH_{-n+5} + (1+i)H_{-n+4} - (1-2i)H_{-n+3} - 2H_{-n+2} - iH_{-n+1} + (1+i)H_{-n}) + i)$.

(b) $\sum_{k=1}^{n} i^k H_{-2k} = \frac{1}{-4+i}(i^n(2H_{-2n+4} - (2+i)H_{-2n+3} - (2-3i)H_{-2n+2} - (1+i)H_{-2n+1} - (5+i)H_{-2n} - H_{-2n-1}) - 1)$.

(c) $\sum_{k=1}^{n} i^k H_{-2k+1} = \frac{1}{-4+i}(i^n(-iH_{-2n+4} + 3iH_{-2n+3} + (1-i)H_{-2n+2} - (3+i)H_{-2n+1} + H_{-2n} + 2H_{-2n-1}) + 2)$.

Taking $H_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$ in the above Proposition, we have the following Corollary which presents sum formulas of Hexanacci-Lucas numbers.

Corollary 5.23. *For $n \geq 1$, Hexanacci-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} i^k E_{-k} = \frac{1}{2+i}(i^n(-iE_{-n+5} + (1+i)E_{-n+4} - (1-2i)E_{-n+3} - 2E_{-n+2} - iE_{-n+1} + (1+i)E_{-n}) + (-8-3i))$.

(b) $\sum_{k=1}^{n} i^k E_{-2k} = \frac{1}{-4+i}(i^n(2E_{-2n+4} - (2+i)E_{-2n+3} - (2-3i)E_{-2n+2} - (1+i)E_{-2n+1} - (5+i)E_{-2n} - E_{-2n-1}) + (20+5i))$.

(c) $\sum_{k=1}^{n} i^k E_{-2k+1} = \frac{1}{-4+i}(i^n(-iE_{-2n+4} + 3iE_{-2n+3} + (1-i)E_{-2n+2} - (3+i)E_{-2n+1} + E_{-2n} + 2E_{-2n-1}) + (-4-2i))$.

Corresponding sums of the other sixth order generalized Hexanacci numbers can be calculated similarly.

6 Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, linear sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written linear sum identities in terms of the generalized Hexanacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Hexanacci, Hexanacci-Lucas, sixth order Pell, sixth order Pell-Lucas, sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth-order Jacobsthal Perrin, adjusted sixth-order Jacobsthal, modified sixth-order Jacobsthal-Lucas, 6-primes, Lucas 6-primes and modified 6-primes sequences. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Natividad LR. On solving fibonacci-like sequences of fourth, fifth and sixth order. International Journal of Mathematics and Computing. 2013;3(2).

[2] Rathore GPS, Sikhwal O, Choudhary R. Formula for finding nth term of Fibonacci-like sequence of higher order. International Journal of Mathematics and Its Applications. 2016;4(2-D):75-80.

[3] Soykan Y. A study on generalized (r,s,t,u,v,y)-numbers. Journal of Progressive Research in Mathematics. 2020;17(1):54-72.

[4] Soykan Y Özmen N. On Generalized Hexanacci and Gaussian Generalized Hexanacci Numbers, Accepted.

[5] Soykan Y. On generalized sixth-order Pell sequence. Journal of Scientific Perspectives. 2020;4(1):49-70.
DOİ: https://doi.org/10.26900/jsp.4.005.

[6] Soykan Y, PolatlıEE. On Generalized Sixth-Order Jacobsthal Sequence, Accepted.

[7] Soykan Y. Properties of generalized 6-primes numbers. Archives of Current Research International. 2020;20(6):12-30.
DOI: 10.9734/ACRI/2020/v20i630199

[8] Sloane NJA. The on-line encyclopedia of integer sequences.
Available:http://oeis.org/

[9] Cook CK, Bacon MR. Some identities for Jacobsthal and Jacobsthal-Lucas numbers satisfying higher order recurrence relations. Annales Mathematicae et Informaticae. 2013;41:27-39.

[10] Akbulak M, Öteleş A. On the sum of Pell and Jacobsthal numbers by matrix Method. Bulletin of the Iranian Mathematical Society. 2014;40(4):1017-1025.

[11] Gökbaş H, Köse H. Some sum formulas for products of Pell and Pell-Lucas numbers. Int. J. Adv. Appl. Math. and Mech. 2017;4(4):1-4.

[12] Öteleş A, Akbulak M. A Note on Generalized k-Pell Numbers and Their Determinantal Representation. Journal of Analysis and Number Theory. 2016;4(2):153-158.

[13] Koshy T. Fibonacci and Lucas Numbers with Applications. A Wiley-Interscience Publication, New York; 2001.

[14] Koshy T. Pell and Pell-Lucas Numbers with Applications, Springer, New York; 2014.

[15] Hansen RT. General Identities for Linear Fibonacci and Lucas Summations, Fibonacci Quarterly. 1978;16(2):121-28.

[16] Soykan Y. On summing formulas for generalized Fibonacci and Gaussian generalized Fibonacci numbers. Advances in Research. 2019;20(2):1-15.

[17] Soykan Y. Corrigendum: On Summing Formulas for Generalized Fibonacci and Gaussian Generalized Fibonacci Numbers, Advances in Research. 2020;21(10):66-82.
DOI: 10.9734/AIR/2020/v21i1030253

[18] Soykan Y. On Summing Formulas for Horadam Numbers. Asian Journal of Advanced Research and Reports. 2020;8(1):45-61.
DOI: 10.9734/AJARR/2020/v8i130192.

[19] Soykan Y. Generalized Fibonacci numbers: Sum formulas. Journal of Advances in Mathematics and Computer Science. 2020;35(1):89-104.
DOI:10.9734/JAMCS/2020/v35i130241.

[20] Soykan Y. Generalized Tribonacci Numbers: Summing Formulas. Int. J. Adv. Appl. Math. and Mech. 2020;7(3):57-76.

[21] Soykan Y. On Sum Formulas for Generalized Tribonacci Sequence. Journal of Scientific Research & Reports. 2020;26(7):27-52.
ISSN: 2320-0227.
DOI: 10.9734/JSRR/2020/v26i730283

[22] Frontczak R. Sums of Tribonacci and Tribonacci-Lucas Numbers. International Journal of Mathematical Analysis. 2018;12(1):19-24.

[23] Parpar T. K'ncı Mertebeden Rekürans Bağıntısının Özellikleri ve Bazı Uygulamaları, Selçuk Üniversitesi, Fen Bilimleri Enstitüsü, Yüksek Lisans Tezi; 2011.

[24] Soykan Y. Summing Formulas for Generalized Tribonacci Numbers. Universal Journal of Mathematics and Applications. 2020;3(1):1-11. ISSN 2619-9653.
DOI: https://doi.org/10.32323/ujma.637876

[25] Soykan Y. Summation Formulas For Generalized Tetranacci Numbers. Asian Journal of Advanced Research and Reports. 2019;7(2):1-12.
doi.org/10.9734/ajarr/2019/v7i230170.

[26] Soykan Y. Matrix Sequences of Tribonacci and Tribonacci-Lucas Numbers. Communications in Mathematics and Applications. 2020;11(2):281-295. DOI: 10.26713/cma.v11i2.1102

[27] Waddill ME. The Tetranacci Sequence and Generalizations, Fibonacci Quarterly. 1992;9-20.

[28] Soykan Y. Sum Formulas for generalized fifth-order linear recurrence sequences. Journal of Advances in Mathematics and Computer Science. 2019;34(5):1-14. Article no.JAMCS.53303. ISSN: 2456-9968.
DOI: 10.9734/JAMCS/2019/v34i530224.

[29] Soykan Y. Linear Summing Formulas of Generalized Pentanacci and Gaussian Generalized Pentanacci Numbers. Journal of Advanced in Mathematics and Computer Science. 2019;33(3):1-14.

[30] Soykan Y. On Summing Formulas of Generalized Hexanacci and Gaussian Generalized Hexanacci Numbers. Asian Research Journal of Mathematics. 2019;14(4):1-14. Article no.ARJOM.50727.

[31] Soykan Y. A Study On Sum Formulas of Generalized Sixth-Order Linear Recurrence Sequences. Asian Journal of Advanced Research and Reports. 2020;14(2):36-48.
DOI: 10.9734/AJARR/2020/v14i230329

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