Logistic Regression without Intercept

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Commentary

Abstract

Logistic regression is a popular statistic modelling algorithm in predicting a binary outcome. Although logistic regression almost always has an intercept, logistic regression without intercept is sometimes appropriate or even necessary. However, logistic regression without intercept has rarely been discussed other than being used explicitly or implicitly. In this paper, we aim at filling this gap by systematically studying logistic regression without intercept. Specifically, we study the 4 most important aspects of logistic regression: (1) Maximum Likelihood Estimate, (2) data configuration (complete separation, quasi-complete separation and overlap) to categorize the existence and uniqueness of maximum likelihood estimate, (3) multicollinearity, and (4) monotonic transformations of independent variables. We adopt an extensional method in that we first present results for logistic regression with intercept and then extend the results to the case of without intercept. Our numerical examples further compare logistic regression with intercept and without intercept.

1 Introduction

Logistic regression and linear regression are two most commonly used regressions for modeling the relationship between the dependent variable and one or more independent variables. By default, both linear regression and logistic regression have an intercept which is a constant term with regard to the coefficients of independent variables. Albeit controversial, linear regression without intercept has been studied over the past 2 decades

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[1,2,3,4]. However, logistic regression without intercept is rarely studied in the literature, although it has been used to study ridge logistic regression [5,6,7,8]. In practice, there are circumstances in which logistic regression without intercept is appropriate or even necessary. For instance, logistic regression without intercept can be used in conditional logistic regression with matched pairs data [9].

In this paper, we study logistic regression without intercept when it is appropriate. Specifically, we derive maximum likelihood estimate (MLE) and its numerical solution, multicollinearity and its detection, characterization of the existence and uniqueness of MLE by data configuration, and linear transformations of independent variables.

The rest of the paper is organized as follows. In Section 2, we introduce logistic regression without intercept, its MLE and numerical solution. In Section 3, multicollinearity is discussed for logistic regression without intercept. In Section 4, three types of data configuration are defined for logistic regression without intercept and used to characterize the existence and uniqueness of MLE by data configuration. Monotonic transformations including linear transformations of independent variables are also discussed. Numerical examples are provided in Section 6. In particular, we show an example which has multicollinearity for logistic regression with intercept but not for the case of without intercept. We also show an example which has separation data configuration for logistic regression with intercept but not for the case of without intercept. Finally, the paper is concluded in Section 7.

Throughout the paper, by logistic regression, we mean logistic regression with intercept unless otherwise specified. We first present the results for logistic regression with intercept and then extend the results to logistic regression without intercept.

2 Maximum Likelihood Estimate

2.1 With intercept

Let us first introduce maximum likelihood estimate (MLE) in logistic regression with intercept. Let $x = (x1, x2, ..., xp)$ be the vector of p independent variables and y the binary dependent variable (also called response or target) with values of 0 and 1. Without loss of generality, we assume $x_1, x_2, ..., x_p$ do not have missing values. Assume we have a sample of N independent observations $(x_{i1}, x_{i2}, ..., x_{ip}, y_i), i = 1, 2, ..., N$, where y_i is the value of y and $x_{i1}, x_{i2}, ..., x_{ip}$ are the values of functions $x_1, x_2, ..., x_p$ for the i-th observation, respectively.

Let X_i be the row vector $(1, x_{i1}, x_{i2}, ..., x_{ip})$ for $i = 1, 2, ..., N$, and denote X the $N \times (p + 1)$ matrix, called Design Matrix, with X_i as rows. To accommodate the intercept, let us denote the N-dimensional column vector with all elements of 1 by x_0. For simplicity of notation, we also let $x_1, x_2, ..., x_p$ stand for the N-dimensional vectors

$$(x_{11}, x_{21}, ..., x_{N1})^T, (x_{12}, x_{22}, ..., x_{N2})^T, (x_{1p}, x_{2p}, ..., x_{Np})^T$$

respectively, when no confusion can arise.

We use $\pi(x) = P(y = 1|x)$ to represent the conditional probability that y is equal to 1 given x. Then the logistic regression model is given by the equation

$$\pi(x) = \frac{e^{\beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p}}{1 + e^{\beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p}} \tag{2.1}$$

where the constants $\beta_0, \beta_1, ... \beta_p$ are called the model parameters, also known as the coefficients corresponding to $x_0, x_1, ... x_p$, respectively. The first constant β_0 is called the intercept.

Note that (2.1) is in the function form. It can be understood in the vector form (dimension N) with i-th component

$$\pi(x_{i1}, x_{i2}, \dots, x_{ip}) = \frac{e^{\beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip}}}{1 + e^{\beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip}}}$$

for $i = 1, 2, \dots, N$.

Equivalently, the logit transformation of $\pi(x)$ in (2.1) is

$$g(x) = \ln\left[\frac{\pi(x)}{1 - \pi(x)}\right] = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p. \tag{2.2}$$

The method of maximum likelihood aims at yielding values for the unknown parameters β_0, $\beta_1, \dots \beta_p$ which maximize the probability of obtaining the observed data. We first construct a so-called likelihood function, which expresses the probability of the observed data as a function of unknown parameters. The contribution of an observation to the likelihood function is $\pi(x_{i1}, x_{i2}, \dots, x_{ip})$ when $y_i = 1$, and $1 - \pi(x_{i1}, x_{i2}, \dots, x_{ip})$ when $y_i = 0$. In other words, we attempt to maximize $\pi(x_{i1}, x_{i2}, \dots, x_{ip})$ when $y_i = 1$ and $1 - \pi(x_{i1}, x_{i2}, \dots, x_{ip})$ when $y_i = 0$. A unified way to express the contribution of an observation to the likelihood function is

$$\pi(x_{i1}, x_{i2}, \dots, x_{ip})^{y_i} [1 - \pi(x_{i1}, x_{i2}, \dots, x_{ip})]^{1-y_i}. \tag{2.3}$$

Since the observations are assumed to be independent, the likelihood function is obtained as the product of all contributions

$$h(X, \beta) = \prod_{i=1}^{N} \pi(x_{i1}, x_{i2}, \dots, x_{ip})^{y_i} [1 - \pi(x_{i1}, x_{i2}, \dots, x_{ip})]^{1-y_i} = \prod_{i=1}^{N} \left(\frac{e^{X_i\beta}}{1 + e^{X_i\beta}}\right)^{y_i} \left(\frac{1}{1 + e^{X_i\beta}}\right)^{1-y_i} \tag{2.4}$$

where β is column vector $(\beta_0, \beta_1, \dots \beta_p)^T$, superscript T is matric transport, and $X_i\beta$ represents matrix multiplication.

Since it is easier to work with the log of equation, the log likelihood is instead used

$$l(\beta) = \sum_{i=1}^{N} y_i \ln\left(\frac{e^{X_i\beta}}{1 + e^{X_i\beta}}\right) + \sum_{i=1}^{N} (1 - y_i) \ln\left(\frac{1}{1 + e^{X_i\beta}}\right). \tag{2.5}$$

The value of β that maximizes $l(\beta)$ in (2.5) is called the maximum likelihood estimate (MLE).

In particular, if $\beta_1 = \beta_2 = \cdots \beta_p = 0$, then it is called the intercept-only model or null model or base model. In this case, no independent variables are considered and $\beta_0 = \ln\left(\frac{\bar{y}}{1 - \bar{y}}\right)$, where $\bar{y} = \frac{\sum_{i=1}^{N} y_i}{N}$.

We can further simplify (2.5) as follows:

$$l(\beta) = \sum_{i=1}^{N} \left[y_i \ln(e^{X_i\beta}) - y_i \ln(1 + e^{X_i\beta})\right] - \sum_{i=1}^{N} (1 - y_i) \ln(1 + e^{X_i\beta}) = \sum_{i=1}^{N} \left[y_i (X_i\beta) - y\ln 1 + eXi\beta - i = 1N1 - y\ln 1 + eXi\beta = i = 1NyiXi\beta - \ln 1 + eXi\beta. \tag{2.6}$$

To maximizing function $l(\beta)$, we differentiate it with respect to β, set the derivative equal to 0. To start with, we take the derivative with respect to one component of β, say, β_j for $j = 0, 1, 2, \dots, p$ to obtain

$$\frac{\partial l}{\partial \beta_j} = \sum_{i=1}^{N} \left(y_i x_{ij} - \frac{e^{X_i\beta}}{1 + e^{X_i\beta}} x_{ij}\right) = \sum_{i=1}^{N} (y_i x_{ij} - \pi_i x_{ij}) \tag{2.7}$$

where $\pi_i = \pi(x_{i1}, x_{i2}, \dots, x_{ip}) = \frac{e^{X_i\beta}}{1 + e^{X_i\beta}}$ for $i = 1, 2, \dots, N$.

Setting (2.7) to 0 for $j = 0, 1, 2, \dots, p$, we obtain

$$\sum_{i=1}^{N} y_i = \sum_{i=1}^{N} \pi_i$$

(2.8)

$$\sum_{i=1}^{N} x_{ij} y_i = \sum_{i=1}^{N} x_{ij} \pi_i, j = 1,2, \dots, p.$$

In general, (2.8) cannot be solved analytically [10,11]. Rather, they can be solved numerically by Newton-Raphson algorithm as follows

$$\beta^{(i+1)} = \beta^{(i)} + (X^T V X)^{-1} g,$$

(2.9)

where V is the $N \times N$ diagonal matrix with its diagonal elements $\pi_1(1 - \pi_1), \pi_2(1 - \pi_2), \dots, \pi_N(1 - \pi_N)$. In addition, $g = X^T(y - \pi)$ with $\pi = (\pi_1, \pi_2, \dots, \pi_N)^T$. Both V and g are evaluated at $\beta^{(i)}$ in (2.9).

2.2 Without intercept

If the intercept is forced out from logistic regression, the parameter vector β will become $(\beta_1, \dots \beta_p)^T$. Vector $x_0 = (1,1, \dots 1)^T$ for the intercept is not needed. Accordingly, row vectors $X_i = (x_{i1}, x_{i2}, \dots, x_{ip})$ for $i = 1, 2, \dots, N$. Moreover, the dimension of design matrix X will be $N \times p$. Equations (2.1) – (2.6) all hold after β_0 is taken out. Again the value of β that maximizes the log likelihood $l(\beta)$ is called the maximum likelihood estimate (MLE).

To maximizing function $l(\beta)$, we differentiate it with respect to β. In this case, (2.7) holds for $j = 1, 2, \dots, p$. Setting (2.7) to 0 for $j = 1, 2, \dots, p$, we obtain the second equation in (2.8), that is,

$$\sum_{i=1}^{N} x_{ij} y_i = \sum_{i=1}^{N} x_{ij} \pi_i, j = 1,2, \dots, p.$$

(2.10)

Clearly, if the original logistic regression model with intercept has a zero intercept $\beta_0 = 0$, logistic regression without intercept will have the same results as logistic regression with intercept (Refer to Example 6.3 in Section 6).

To get a numerical solution to the p simultaneous nonlinear equations of β in (2.10), Newton-Raphson in the form of (2.9) can be used.

3 Multicollinearity

3.1 With intercept

Perfect Multicollinearity or Complete Multicollinearity or Multicollinearity in short refers to a situation in logistic regression in which two or more independent variables are linearly related [12,13,14]. Mathematically, multicollinearity means there exist constant $a_0, a_1, \dots, a_p$ such that

$$\sum_{i=0}^{p} a_i x_i = 0$$

(3.1)

where at least two of $a_1, \dots, a_p$ are nonzero. If we treat x_0 as an independent variable, then we just require at least one of $a_1, \dots, a_p$ is nonzero.

In particular, if two independent variables are linearly related, then it is called collinearity. Some authors use near multicollinarity [15] and partial multicollinearity [14] when there is an approximate linear relationship among two or more independent variables.

Some authors define multicollinearity to be high correlation between independent variables [3,16,17]. Correlation indicates the linear relationship between a pair of independent variables. It can be measured by the

well-known Pearson correlation coefficient [18]. The perfect linear relationship is indicated by a correlation coefficient of -1 or 1.

If there is multicollinearity, the design matrix X will not have a full column rank of $p + 1$. Hence, the $(p + 1) \times (p + 1)$ matrix $\hat{I} = X^T V X$ in (2.9) will have a rank less than $p + 1$. Thus, the inverse matrix $\hat{I}^{-1}$ in (2.9) does not exist, which make the iteration in (2.9) impossible.

If there is near multicollinearity and there is no separation of the data points, theoretically $\hat{I} = X^T V X$ in (2.9) has an inverse and the iteration in (2.9) can be proceeded. Yet, iteration (2.9) may not find an approximate inverse $\hat{I} = X^T V X$ and hence may cause unstable estimates and inaccurate variances [19].

Multicollinearity can be easily detected by using variance inflation factor (VIF). Specifically, the VIF is calculated for each independent variable by doing a linear regression of that independent variable on all the other independent variables, and then obtaining the R^2 (the coefficient of determination) from the regression. The VIF for this independent variable is just $1/(1-R^2)$. A VIF of 1 means that there is no correlation among this independent variable and the remaining independent variables. The larger the VIF of an independent variable, the larger correlation between this independent variable and others. There is no standard for acceptable levels of VIF.

Multicollinearity can be also detected by correlation coefficient. However, correlation coefficient is used between two independent variables, whereas VIF can be used to check the linear relationship of one independent variable with all other independent variables. In this sense, VIF is preferred.

3.2. Without intercept

For logistic regression without intercept, we modify the definition of multicollinearity in (3.1) by removing constant a_0. Clearly, if there exists multicollinearity in logistic regression without intercept, there exists multicollinearity in logistic regression with intercept too. Example 6.4 in Section 6 shows that we cannot keep constant a_0 in (3.1) in the case of without intercept.

To calculate VIF, we do a linear regression without intercept for each independent variable on all the other independent variables. We then adjust R^2 [2] as

$$R^2(\beta_j) = \frac{\sum_{i=1}^N \hat{x}_{ij}^2}{\sum_{i=1}^N x_{ij}^2}, j = 1,2,\dots,p$$

where $\hat{x}_{ij}$ is the is the i-th fitted value. Finally, the VIF for x_j will be $\frac{1}{\left(1-R^2(\beta_j)\right)}$. Note that correlation coefficient does not necessarily indicate multicollinearity [3] in the case of without intercept.

4 Data configuration

4.1 With intercept

Albert and Anderson [20] first assumed design matrix X to have a full column rank, that is, no multicollinearity. He then introduced the concept of separation (including complete separation and quasi-complete separation) and overlap in logistic regression with intercept. He showed that separation leads to nonexistence of (finite) MLE and that overlap leads to finite and unique MLE.

With slight modifications to the definition of pseudo-complete separation, the concept of separation can be applied to and design matrices. Of course, the assumption of full column rank will need to be added to overlap in order for MLE to exist and to be unique.

Definition 4.1. There is *quasi-complete separation* if the data are not complete separable, but there exists a non-zero vector $b = (b_0, b_1, \dots, b_p)^T$ such that

$$\begin{cases} \sum_{j=0}^{p} b_j x_{ij} = X_i b = b^T X_i^T \geq 0, \ y_i = 1, \\ \sum_{j=0}^{p} b_j x_{ij} = X_i b = b^T X_i^T \leq 0, \ y_i = 0 \end{cases} \tag{4.1}$$

and equality holds for at least one subject in each response group but not for all $i = 1, 2, \ldots, N$.

Remark 4.2. Zeng [21] pointed out variants of complete separation and quasi-complete separation.

4.2 Without intercept

In the case of without intercept, we adjust the definition of complete separation in Albert and Anderson [20] and Definition 4.1 above by removing constant b_0 from $b = (b_0, b_1, \ldots, b_p)^T$ and changing the lower limit from $j = 0$ to $j = 1$ in the summations. The definition of overlap remains the same.

Clearly, if there is a separation (complete or pseudo-complete) in logistic regression without intercept, there will be a separation in logistic regression with intercept.

Similar to Albert and Anderson [20], we can completely characterize the existence and uniqueness of MLE in logistic regression without intercept as follows.

Theorem 4.3. For logistic regression without intercept,

(i) if there is complete separation, then MLE does not exist;
(ii) if there is quasi-complete separation, then MLE does not exist;
(iii) If there is overlap and X has a full column rank of p, then MLE exists and is unique.

In the following, we will prove (ii). As to (i) and (iii), they can be proved similarly to those in Zeng [22] for weighted logistic regression by taking all weights = 1.

Proof of Theorem 4.3. (ii). Assume $b = (b_1, b_2, \ldots, b_p)^T$ satisfies (4.1). Let $\beta(k) = kb$ for $k > 0$. Then

$$l(\beta(k)) = \sum_{i=1}^{r} ln\left(\frac{e^{kX_i b}}{1 + e^{kX_i b}}\right) + \sum_{i=r+1}^{n} ln\left(\frac{1}{1 + e^{kX_i b}}\right) = \sum_{i=1}^{r} ln\left(1 - \frac{1}{1 + e^{kX_i b}}\right) + \sum_{i=r+1}^{n} ln\left(\frac{1}{1 + e^{kX_i b}}\right).$$

Since there is at least one i with $1 \leq i \leq r$ such that $X_i b > 0$ or one i with $r + 1 \leq i \leq N$ such that $X_i b < 0$. If $X_i b = 0$ for $1 \leq i \leq N$, both $ln\left(1 - \frac{1}{1 + e^{kX_i b}}\right)$ and $ln\left(\frac{1}{1 + e^{kX_i b}}\right)$ have a constant value $- ln(2)$. If $X_i b > 0$ for $1 \leq i \leq r$, then $ln\left(1 - \frac{1}{1 + e^{kX_i b}}\right)$ is strictly increasing in k. If $X_i b > 0$ for $r + 1 \leq i \leq N$, then $ln\left(\frac{1}{1 + e^{kX_i b}}\right)$ is strictly increasing in k.

Therefore,

$$l(\beta) = l(\beta(k = 1)) < L(\beta(2)) < \cdots \leq \lim_{k \to \infty} l(\beta(k)).$$

The maximum likelihood estimate $\&$ is thus at infinity on the boundary of R^p.

5 Monotonic Transformations

In data analysis, a transformation is a mapping of a variable into a new variable by a function of the original variable. A transformation can be linear or nonlinear, depending on whether the function is linear or nonlinear. A monotonic transformation is a transformation whose function is monotonic (increasing or decreasing). Clearly, a linear transformation is a monotonic transformation.

Sometimes, linear transformation is used to transfer one independent variable to a new independent variable. For instance, linear transformation can be used for the purpose of scale [23]. Note that we don't consider the trivia case – the constant transformation, as a variable with a constant value will not provide any information [24]. If

coefficient β for variable x is too small, we may make the coefficient larger by making a linear transformation $x' = \frac{x}{\alpha}$ with $\alpha > 1$, without changing the model. For instance, if $\alpha = 100$, then the coefficient with transformed variable x' will be 100 times as large as β.

The following result shows that linear transformations do not change logistic regression model.

Theorem 5.1. Logistic regression with intercept is invariant under non-constant linear transformations of independent variables.

Proof. Assume we make linear transformations for k independent variables, where $k \leq p$. Without loss of generality, we assume $x_1, x_2, \ldots, x_k$ are such k independent variables. Let the k linear transformations be $x_i' = a_i x_i + b_i$, where $a_i \neq 0$ for $i = 1, 2, \ldots, k$. Then

$$\beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p$$
$$= \beta_0 + \beta_1\left(\frac{1}{a_1}x_1' - \frac{1}{b_1}\right) + \beta_2\left(\frac{1}{a_2}x_2' - \frac{1}{b_2}\right) + \cdots + \beta_k\left(\frac{1}{a_k}x_k' - \frac{1}{b_k}\right) + \beta_{k+1}x_{k+1} + \cdots + \beta_p x_p$$
$$= \beta_0 - \left(\frac{\beta_1}{b_1} + \frac{\beta_2}{b_2} + \cdots + \frac{\beta_k}{b_k}\right) + \frac{\beta_1}{a_1}x_1' + \frac{\beta_2}{a_2}x_2' + \cdots + \frac{\beta_k}{a_k}x_k' + \beta_{k+1}x_{k+1} + \cdots + \beta_p x_p.$$

By Albert (1984), the MLE is unique if it exists. If $\beta_0, \beta_1, \ldots \beta_p$ is the MLE with independent variables $x_1, x_2, \ldots, x_p$, then $\beta_0 - \left(\frac{\beta_1}{b_1} + \frac{\beta_2}{b_2} + \cdots + \frac{\beta_k}{b_k}\right), \frac{\beta_1}{a_1}, \frac{\beta_2}{a_2}, \ldots, \frac{\beta_k}{a_k}, \beta_{k+1}, \ldots, \beta_p$ is the MLE with independent variables $x_1', x_2', \ldots, x_k', x_{k+1}, \ldots, x_p$. Moreover, the probabilities $\pi(X_i), i = 1, 2, \ldots, N$ are identical with independent variables $x_1', x_2', \ldots, x_k', x_{k+1}, \ldots, x_p$ are identical to those with independent variables $x_1, x_2, \ldots, x_p$.

Similarly, logistic regression without intercept is invariant under linear transformations without constant.

Theorem 5.2. Logistic regression without intercept is invariant under linear transformations without constant of independent variables.

From the proof of Theorem 5.1, we see that logistic regression without intercept is not invariant under linear transformations with constant unless $\beta_0 = \left(\frac{\beta_1}{b_1} + \frac{\beta_2}{b_2} + \cdots + \frac{\beta_k}{b_k}\right)$. See Example 6.2 in Section 6.

Zeng [21] showed that MLE, information value [24], mutual information [25], Kolmogorov–Smirnov (KS) [26] statistics regarding a univariate logistic regression are all invariant under monotonic transformations. Following the same arguments, we can prove that these properties hold for univariate logistic regression without intercept.

6 Numerical Examples

In this section, we conduct comparison between logistic regression models with intercept and without intercept by numerical examples of the real world.

We use the statistical software package R (version 3.4.2) and its RStudio for our numerical examples.

We use the German Credit Data from a German bank. They are hosted by the UCI Machine Learning Repository [27]. The Germany data contains 1000 observations for 1000 loan applicants. The dependent variable is credit_status: 1 for good loans and 2 for bad loans. For the purpose of logistic regression, we define a new variable called default as default = credit_status − 1. With the new variable default, 0 is for good loans and 1 is for bad loans.

We will use the 3 following continuous independent variables to run logistic regression:

- duration: Duration in month
- age: Age in years
- num_credits: Nmber of existing credits at this bank.

As we have only 1000 observations, we don't split the data into training and test but use all the 1000 observations. Let's name the Germany Credit Data germany_credit. We italicize the text for the R code and use '>' as prompt and '#' for comments. Since we are not concentrating on model development, we shall not pay attention to the significance (P-value) of the independent variables.

Example 6.1.

Let's call the original logistic regression model with intercept by Model 1.

> model1 <- glm (default ~ age + duration + num_credits, data = german_credit, family = binomial)
The coefficients can be found by coefficients(model1) as follows:

(Intercept)   age   duration num_credits
-0.87562523 -0.01738049 0.03747538 -0.12979943

Now let's predict the probabilities of the first 5 rows.

> predict(model1, newdata = german_credit[1:5,], type='response')

          1   2   3   4   5
    0.1115639 0.6013331 0.1966501 0.4468069 0.2392163

Now let's force no intercept. This can be done by subtracting 1 from or adding 0 to the list of independent variables as follows. Let's call the logistic regression model without intercept by Model 2.

> model2 <- glm (default ~ age + duration + num_credits - 1, data = german_credit, family = binomial)

The coefficients are as follows:

          age   duration num_credits
-0.03034242 0.03126543 -0.31047647

Now let's predict the probabilities for the first 5 records.

> predict(model2, newdata = german_credit[1:5,], type='response')

          1   2   3   4   5
0.07825452 0.62778616 0.19433730 0.41029713 0.18561920

We see that none of the first 5 records have the same probability as logistic regression with intercept. Moreover, the order of the probabilities has changed. The 5th record has the 4th largest probability in the model without intercept, whereas the 3rd record has the 3rd largest probability in the model with intercept. Indeed, KS and ACU of these 2 models are different as we can see below.

We apply a user-defined R function called KS_AUC (see the appendix).

> predicted1<- predict(model1, newdata = german_credit, type='response')
> predicted1 = cbind(german_credit, predicted1)
> predicted1$score = round   (1000 * (1 - predicted1$predicted))
> KS_AUC(predicted1, score, default)

KS = 0.207142857142857

Area under the curve: 0.642

> predicted2 <- predict(model2, newdata = german_credit, type='response')
> predicted2 = cbind(german_credit, predicted2)

> predicted2$score = round   (1000 * (1 - predicted2$predicted))
> KS_AUC(predicted2, score, default)

KS = 0.192857142857143

Area under the curve: 0.633

We conclude that the logistic regression model without intercept is different from the logistic regression model with intercept.

Example 6.2

Let's make a linear transformation for age, force no intercept and call the logistic regression model by Model 3.

> german_credit$age1 = 2 * german_credit$age - 1
> model3 <- glm (default ~ age1 + duration + num_credits - 1, data = german_credit, family = binomial)

The coefficients are as follows:

```
        age1      duration   num_credits
-0.01522484   0.03110889   -0.31561646
```

Now let's predict the probabilities of the first 5 rows.

> predict(model3, newdata = german_credit[1:5,], type='response')
```
          1           2           3           4           5
0.07802539  0.62783546   0.19479968   0.40997857   0.18494037
```

They are different from but closed to those in model2.

Example 6.3.

Let's make a linear transformation for age to age1, and call the logistic regression model by Model 4.

> german_credit$age1 = 2 * german_credit$age - 1
> model4 <- glm (default ~ age1 + age + duration + num_credits, data = german_credit, family = binomial)

The coefficients are as follows:

```
(Intercept)            age1 age       duration    num_credits
-0.884315473  -0.008690246  NA   0.037475381   -0.129799431
```

We see that the coefficient for age is missing. R automatically keeps only one of age1 and age as they are collinear.

Indeed, if we let $x_1 = age$, and $x_2 = age1$, then (3.1) holds with $a_0 = 1, a_1 = -2$ and $a_2 = 1$. Therefore, age and age1 are collinear.

Multicollinearity can be checked using function vif in the car package in R. In our case, an error message "there are aliased coefficients in the model" will show up, which indicates multicollinearity in the logistic regression model. In this context, one variable is an "alias" of another variable, that is, one variable is linearly dependent on another variable. Since age1 and age are collinear, $R^2 = 1.$ When one does linear regression of age vs age1 or age1 vs age, the VIF of age or age1 is not defined as the denominator in $vif = 1/(1 - R^2)$ is 0.

Let's call the logistic regression model without intercept by Model 5:

> model5 <- glm (default ~ age1 + age + duration + num_credits - 1, data = german_credit, family = binomial)

The coefficients are as follows:

```
    age1          age       duration  num_credits
 0.87562523  -1.76863095   0.03747538  -0.12979943
```
Both age and age1 are included in the model. They are not collinear as age1 and age have a linear relationship with constant. Hence, we have an example which has multicollinearity in logistic regression with intercept but not in logistic regression without intercept.

Example 6.4.

Let's define a new variable x as follows: For $i = 1, 2, \ldots, 1000$,

$$x_i = \begin{cases} \min_{y_i=1}(age_i) + 50, & if \ y_i = 1 \\ \max_{y_i=0}(age_i) + 50 & if \ y_i = 0. \end{cases}$$

Then for $i = 1, 2, \ldots, 1000$,

$$\begin{cases} 50 + age_i - x_i \geq 0, \ y_i = 1; \\ 50 + age_i - x_i \leq 0, \ y_i = 0. \end{cases}$$

Moreover, $50 + age_i - x_i = 0$ when age_i attains the minimum value among those i's such that $y_i = 1$, and $50 + age_i - x_i = 0$ when age_i attains the maximum value among those i's such that $y_i = 0$. Hence, quasi-complete separation exists for both logistic regression with intercept and without intercept as we can see below.
> german_credit$x<- ifelse(german_credit$default == 1,min(german_credit$age[german_credit$default==1] + 50),max(german_credit$age[german_credit$default == 0] + 51))

> model6 <- glm (default ~ x + age + duration + num_credits, data = german_credit, family = binomial)

However, overlap exists for logistic regression without intercept.

> model7 <- glm (default ~ x + age + duration + num_credits - 1, data = german_credit, family = binomial)

Hence, we have an example which has data configuration of separation in logistic regression with intercept but not in logistic regression without intercept. The reason is that quasi-complete separation does not exist for logistic regression without intercept as constant 50 is included.

7 Conclusions

In this paper, we have studied logistic regression without intercept. We first derived the maximum likelihood estimate (MLE) for logistic regression without intercept and its numerical solution. We then extended the definition of multicollinearity for logistic regression with intercept to the case of without intercept. Next, we extended three types of data configuration (complete separation, quasi-complete separation and overlap) for logistic regression with intercept to the case of without intercept, and characterized the existence and uniqueness of MLE. In addition, we discussed monotonic transformations including linear transformations of independent variables in logistic regression without intercept. Our numerical examples further compared logistic regression with intercept and without intercept.

Competing Interests

Author has declared that no competing interests exist.

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Appendix

```
KS_AUC <- function(df, score, target) {
  score <- deparse(substitute(score))
  target <- deparse(substitute(target))
  sample1 <- df[[score]][which(df[[target]] == 0)]
  sample2 <- df[[score]][which(df[[target]] == 1)]
  cdf1 <- ecdf(sample1)
  cdf2 <- ecdf(sample2)
  minMax <- seq(min(sample1, sample2), max(sample1, sample2), length.out = length(sample2))
  x0 <- minMax[which(abs(cdf1(minMax) - cdf2(minMax)) == max(abs(cdf1(minMax) -
              cdf2(minMax))))]

  y0 <- cdf1(x0)
  y1 <- cdf2(x0)
  KS <- y1 - y0
  message(paste('KS = ', KS))

  # Install pROC package if not.
  # install.packages("pROC")
  library(pROC)
  auc(roc(df[[target]], df[[score]]))
}
```