

Research Article

Analytical Solution for the Gross-Pitaevskii Equation in Phase Space and Wigner Function

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In this work, we study symplectic unitary representations for the Galilei group. As a consequence a nonlinear Schrödinger equation is derived in phase space. The formalism is based on the noncommutative structure of the star product, and using the group theory approach as a guide a physically consistent theory is constructed in phase space. The state is described by a quasi-probability amplitude that is in association with the Wigner function. With these results, we solve the Gross-Pitaevskii equation in phase space and obtained the Wigner function for the system considered.

1. Introduction

A relevant equation that describes a variety physical phenomena, as a Bose-Einstein condensed, is the Gross-Pitaevskii equation [1]. The Gross-Pitaevskii model is an extension of the Schrödinger equation, and it is given by [2, 3]

$$i \frac{\partial \psi(r, t)}{\partial t} = \left[-\frac{1}{2m} \nabla^2 + V_{\text{ext}} + g |\psi(r, t)|^2 \right] \psi(r, t), \quad (1)$$

where m represents mass, V_{ext} is the interaction potential, and g is the intensity of atomic interaction. The study of solutions for this equation is relevant in both ways, theoretical and applied viewpoints. In addition, an important case for the Gross-Pitaevskii system is their approach in quantum phase space, particularly the calculation of Wigner function for this system, in which it is not known in the literature.

In this context, the first formalism to quantum mechanics is phase space which was introduced by Wigner notion of phase space in 1932 [4]. He was motivated by the problem of finding a way to improve the quantum statistical mechanics. Wigner introduced his formalism by using a kind of Fourier transform of the density matrix, $\rho(q, q')$, giving rise

to what is nowadays called the Wigner function, $f_W(q, p)$, where (q, p) are the coordinates of a phase space manifold (Γ) [4–7]. The Wigner function has the same content of usual wave function obtained by the Schrödinger equation. However, the Wigner function is identified as a quasiprobability density in the sense that $f_W(q, p)$ is real but not positively definite and as such cannot be interpreted as a probability. However, the integrals $\rho(q) = \int f_W(q, p) dp$ and $\rho(p) = \int f_W(q, p) dq$ are (true) distribution functions. The calculation of Wigner function is based in the following steps: (i) first, the Schrödinger equation for a specific potential must be solved; (ii) in sequence, using the solutions founded, the matrix density elements are calculated; (iii) finally, a kind of Fourier transform of matrix elements must be performed. We notice that in the Wigner approach it is complicated to treat nonlinear potentials such as the Gross-Pitaevskii system. For this reason, other methods to quantum mechanics in phase space are developed in the literature. There is an alternative method based on the following property of Wigner formalism: in the Wigner function approach, each operator, A , defined in the Hilbert space, \mathcal{H} , is associated with a function, $a_W(q, p)$, in Γ . This procedure is precisely specified by a mapping $\Omega_W : A \rightarrow a_W(q, p)$, such

that, the associative algebra of operators defined in \mathcal{H} turns out to be an algebra in Γ , given by $\Omega_W : AB \longrightarrow a_W \star b_W$, where the star product, \star , is defined by

$$a_W \star b_W = a_W(q, p) \exp \left[\frac{i\hbar}{2} \left(\frac{\overleftarrow{\partial}}{\partial q} \frac{\overleftarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p} \frac{\overleftarrow{\partial}}{\partial q} \right) \right] b_W(q, p). \quad (2)$$

The result is a noncommutative structure in Γ that has been explored in different ways [5–27].

Using star operators defined in Wigner formalism, unitary representations of symmetry Lie groups have been developed on a symplectic [28–32]. The unitary representation of Galilei group leads to the Schrödinger equation in phase space. In the analog procedure, the scalar Lorentz group for spin 0 and spin 1/2 leads to the Klein-Gordon and Dirac equations in phase space. In both cases, relativistic and non-relativistic, the wave functions are closely associated with the Wigner function [28, 29]. In terms of nonrelativistic quantum mechanics, the proposed formalism has been used to treat a nonlinear oscillator perturbatively, to study the notion of coherent states and to introduce a nonlinear Schrödinger equation from the point of view of phase space. In this context, there are a few examples of analytical solutions such as the harmonic oscillator [33], the Hydrogen atom [34], and some spin systems [35–37]. In the present work, we apply this symplectic formalism to find the Wigner function for the Gross-Pitaevskii model. We find an analytical solution for the wave function but the Wigner function is calculated up to a given order of approximation of the star product.

The paper is organized as follows: in Section 2, we write the nonlinear equation in phase space and we present the relation between phase space amplitude and Wigner function. In Section 3, we solve the Gross-Pitaevskii equation in phase space and calculate the Wigner function. In Section 4, we plot graphs of Wigner function and calculate nonnegativity parameter associated to the system. Finally, some closing comments are given in Section 5.

2. Nonlinear Schrödinger Equation in Phase Space

Using the star operators, $\hat{A} = a(q, p) \star$, we define the momentum and position operators, respectively, by

$$\hat{Q} = q \star = q + \frac{i\hbar}{2} \partial_p, \quad (3)$$

$$\hat{P} = p \star = p - \frac{i\hbar}{2} \partial_q. \quad (4)$$

Then, we introduce the following operators:

$$\hat{K} = m\hat{Q}_i - t\hat{P}_i, \quad (5)$$

$$\begin{aligned} \hat{L}_i &= \epsilon_{ijk} \hat{Q}_j \hat{P}_k = \epsilon_{ijk} q_j p_k - \frac{i\hbar}{2} \epsilon_{ijk} q_j \frac{\partial}{\partial p_k} \\ &\quad + \frac{i\hbar}{2} \epsilon_{ijk} p_k \frac{\partial}{\partial q_j} + \frac{\hbar^2}{4} \frac{\partial^2}{\partial q_j \partial p_k} \end{aligned} \quad (6)$$

$$\begin{aligned} \hat{H} &= \frac{\hat{P}^2}{2m} = \frac{1}{2m} (\hat{P}_1^2 + \hat{P}_2^2 + \hat{P}_3^2) \\ &= \frac{1}{2m} \left[\left(p_1 - \frac{i\hbar}{2} \frac{\partial}{\partial q_1} \right)^2 + \left(p_2 - \frac{i\hbar}{2} \frac{\partial}{\partial q_2} \right)^2 + \left(p_3 - \frac{i\hbar}{2} \frac{\partial}{\partial q_3} \right)^2 \right]. \end{aligned} \quad (7)$$

These operators, given in Equations (3), (4), (5), and (6), are defined in the Hilbert space, $\mathcal{H}(\Gamma)$, constructed with complex functions in the phase space [28], and satisfy the set of commutation relations for the Galilei-Lie algebra, that is,

$$\begin{aligned} [\hat{L}_i, \hat{L}_j] &= i\hbar \epsilon_{ijk} \hat{L}_k, \\ [\hat{L}_i, \hat{K}_j] &= i\hbar \epsilon_{ijk} \hat{K}_k, \\ [\hat{L}_i, \hat{P}_j] &= i\hbar \epsilon_{ijk} \hat{P}_k, \\ [\hat{K}_i, \hat{P}_j] &= i\hbar m \delta_{ij}, \\ [\hat{K}_i, \hat{H}] &= i\hbar \hat{P}_i, \end{aligned} \quad (8)$$

with all other commutation relations being null. This is the Galilei-Lie algebra with a central extension characterized by m . The operators defining the Galilei symmetry \hat{P} , \hat{K} , \hat{L} , and \hat{H} are the generators of translations, boost, rotations, and time translations, respectively. \hat{Q} and \hat{P} can be taken to be the physical observable of position and momentum. To be consistent, generators \hat{L} are interpreted as the angular momentum operator, and \hat{H} is taken as the Hamiltonian operator. The Casimir invariants of the Lie algebra are given by

$$\begin{aligned} I_1 &= \hat{H} - \frac{\hat{P}^2}{2m}, \\ I_2 &= \hat{L} - \frac{1}{m} \hat{K} \times \hat{P}, \end{aligned} \quad (9)$$

where I_1 describes the Hamiltonian of a free particle and I_2 is associated with the spin degrees of freedom. First, we study the scalar representation, i.e. spin zero.

Using the time-translation generator, \hat{H} , we derive the time-evolution equation for $\psi(q, p, t)$, i.e.,

$$i\hbar \partial_t \psi(q, p; t) = H(q, p) \star \psi(q, p; t), \quad (10)$$

which is the Schrödinger equation in phase space [28]. The function $\psi(q, p, t)$ is defined in a Hilbert space $\mathcal{H}(\Gamma)$ associated to phase space Γ [28].

The association of $\psi(q, p, t)$ with the Wigner function is given by [28]

$$f_W(q, p) = \psi(q, p, t) \star \psi^\dagger(q, p, t). \quad (11)$$

This function satisfies the Liouville-von Neumann equation [28]. This provides a complete set of physical rules to interpret representations and opens the way to study other improvements. In this sense, the nonlinear Schrödinger equation in phase space is given by

$$\begin{aligned} i \frac{\partial}{\partial t} \psi(q, p, t) = & \left(\frac{p^2}{2m} - \frac{\hbar^2}{8m} \frac{\partial^2}{\partial q^2} - i\hbar \frac{p}{2m} \frac{\partial}{\partial q} \right) \psi(q, p, t) \\ & + V \left(q + \frac{i\hbar}{2}, t \right) \psi(q, p, t) \\ & + g |\psi(q, p, t)|^2 \psi(q, p, t), \end{aligned} \quad (12)$$

where g is intensity of atomic interaction. This equation describes several physical systems; in particular, it is used to study the Bose-Einstein condensation. Equation (20) is derived from the Lagrangian density

$$\begin{aligned} \mathcal{L} = & \frac{i\hbar}{2} (\psi^\dagger \partial_t \psi - \psi \partial_t \psi^\dagger) + \frac{i\hbar}{4m} p (\psi^\dagger \partial_q \psi - \psi \partial_q \psi^\dagger) \\ & - \frac{p^2}{2m} \psi \psi^\dagger + V(q) \star (\psi \psi^\dagger) - \frac{\hbar^2}{8m} \partial_q \psi \partial_q \psi^\dagger + g \psi^4. \end{aligned} \quad (13)$$

In the next section, we solve this equation and then calculate the Wigner function for this system by expanding the star product up to the second order of approximation.

3. Solution of Gross-Pitaevskii Equation and Wigner Function

In this section, we present a solution for the nonlinear Schrödinger equation in phase space and the associated Wigner function.

In order, the Schrödinger equation in phase space is written by

$$\begin{aligned} i \frac{\partial}{\partial t} \psi(q, p, t) = & \left(\frac{p^2}{2m} - \frac{\hbar^2}{8m} \frac{\partial^2}{\partial q^2} - i\hbar \frac{p}{2m} \frac{\partial}{\partial q} \right) \psi(q, p, t) \\ & + V_{\text{ext}} \left(q + \frac{i\hbar}{2}, t \right) \psi(q, p, t) \\ & + g (\psi(q, p, t) \psi^\dagger(q, p, t)) \psi(q, p, t). \end{aligned} \quad (14)$$

In this work, we address the stationary equation without external potential, i.e., $V_{\text{ext}} = 0$. In this way, the equation above becomes

$$\begin{aligned} \left(\frac{p^2}{2m} - \frac{\hbar^2}{8m} \frac{\partial^2}{\partial q^2} - i\hbar \frac{p}{2m} \frac{\partial}{\partial q} \right) \psi(q, p) + g (\psi(q, p) \psi^\dagger \\ \cdot (q, p)) \psi(q, p) = E \psi(q, p). \end{aligned} \quad (15)$$

We now consider the solution of Equation (15) in the regions of constant potential, which may be taken to be $V(\bar{q}) = 0$ without loss of generality. We note first that if $\psi(q, p)$ vanishes anywhere in an interval, as for example at the edges of the box, then $\psi(q, p)$ may be taken to be purely real throughout that interval. This can be done only if

$$\frac{\partial \psi(q, p)}{\partial q} \ll \frac{\partial^2 \psi(q, p)}{\partial q^2}. \quad (16)$$

Thus, we may remove the absolute value symbol in Equation (15). So letting $m = \hbar = 1$, the equation becomes an ordinary nonlinear equation for a real function:

$$\frac{1}{2} \left(p^2 - \frac{1}{4} \frac{\partial}{\partial q} \right) \psi(q, p) + g \psi(q, p)^3 - E \psi(q, p) = 0. \quad (17)$$

Letting $g > 0$, the solution of Equation (17) is

$$\psi(q, p) = A \operatorname{sn} \left(\sqrt{k^2 - p^2} q + \delta \mid m \right), \quad (18)$$

with $k > p$, $\operatorname{sn}(x \mid m)$ is the Jacobian elliptic function, and k and δ will be determined by the boundary conditions, while A and m will be determined by substituting (18) into (17) and normalization. The boundary conditions

$$\psi(0, p) = \psi(L, p) = 0. \quad (19)$$

The boundary condition at the origin can be satisfied by taking $\delta = 0$. The function $\operatorname{sn}(x \mid m)$ is periodic in x with a period of $4K(m)$, where $K(m)$ is the elliptic integral of the first kind. Thus, the boundary equations are satisfied if $k = 2nK(m)/L$, where $n = 1, 2, 3, \dots$. The number of nodes in the solution n th is $n - 1$. We then solve Equation (17) substituting (18), and using the Jacobian elliptic identities, this results in the equation for the amplitude A and energy E ,

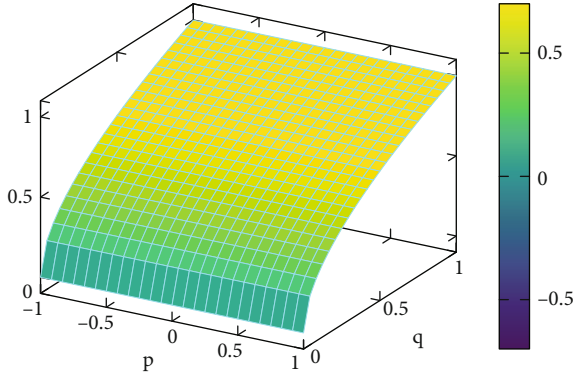
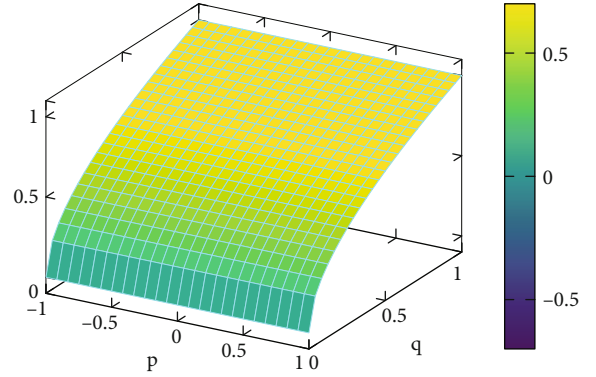
$$A^2 = \frac{2m[2nK(m)]^2}{L^2}, \quad (20)$$

$$E = \frac{[2nK(m)]^2}{2L^2} (m + 1). \quad (21)$$

Equation (18) becomes

$$\psi(q, p) = \frac{\sqrt{2}\sqrt{2m}(2nK(m))}{L} \operatorname{sn} \left(\sqrt{(2nK(m))^2 - p^2} q \mid m \right), \quad (22)$$

The wave-function and the energy are determined up to factor m . This result is similar to what is obtained in

FIGURE 1: Wigner function for Gross-Pitaevskii model; $n = 100$.FIGURE 2: Wigner function for Gross-Pitaevskii model; $n = 200$.

configuration space [1]. Using Equation (22), we calculate Wigner function for the Gross-Pitaevskii system by Equation (11). There are in fact some different approaches from our proposal to calculate the Wigner function, such as the use of the parity operator for this purpose [38, 39]. In this article, the Wigner function is calculated up to the second order in the expansion of the star product. In the next section, this function under these terms is analyzed.

4. Analysis of Solution

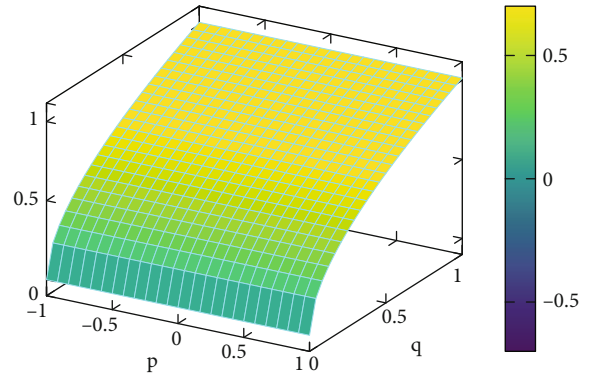
In this section, we plot the graphics to Wigner function founded in the section above and calculate the negativity parameter for this system.

4.1. Particle in a Box Limit. Both the zero density linear limit and the highly excited-state limit give the particle in a box limit type solution. Mathematically, $m \rightarrow 0^+$ and $\sin \rightarrow \sin$. Physically, $n \gg 1$. In this limit, $K(m) \rightarrow \pi[1/2 + m/8 + O(m^2)]$ and $m \rightarrow 1/n^2\pi^2$, so Equation (21) becomes

$$E = \frac{n^2\pi^2}{2L^2} \left(1 + \frac{3m}{2} + O(m^2) \right), \quad (23)$$

which converges to the linear quantum mechanics particle in a box as $m \rightarrow 0^+$. One may obtain these results from the first order perturbation theory [31]. The behavior of the Wigner function given in Equation (22) for the three first energy levels can be visualized in Figures 1–3.

Using the Wigner function, the negative parameter for the system is calculated. The results are presented in Table 1. As the negativity parameter is correlated with the nonclassicality of the system, it seems that in the limit where $n \gg 1$ the BCE behaviors are in accordance with classical mechanics which is a consistent result. It is important to note that such classic behavior may be a consequence of the second order expansion of the star product. The investigation of a general solution is necessary to understand the role of the negativity parameter. Although the expansion of the star product is a legitimate procedure given the order of magni-

FIGURE 3: Wigner function for Gross-Pitaevskii model; $n = 300$.TABLE 1: Parameter of negativity for $n = 100, 200,$ and 300 .

n	$\eta(\psi)$
100	0
200	0
300	0

tude of the Planck constant, in a future work, we hope to apply a more precise method to understand which order of expansion is ideal.

5. Concluding Remarks

In this work, we studied the nontrivial problem of the Gross-Pitaevskii equation in phase space, in which is a case of nonlinear Schrödinger. The Wigner function for this system was obtained. In our knowledge, this is the first time that an analytical solution of the Wigner function for the Gross-Pitaevskii system appears in the literature. The particle in a box solution is studied for the physical meaningful limit of the solution, $n \gg 1$. We studied the parameter of negativity of the system and concluded that in these limits it appears to have a purely classical behavior.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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