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Research Article

Fractional Operators Associated with the p-Extended Mathieu Series by Using Laplace Transform

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In this paper, our leading objective is to relate the fractional integral operator known as P_δ-transform with the p-extended Mathieu series. We show that the P_δ-transform turns to the classical Laplace transform; then, we get the integral relating the Laplace transform stated in corollaries. As corollaries and consequences, many interesting outcomes are exposed to follow from our main results. Also, in this paper, we have converted the P_δ-transform into a classical Laplace transform by changing the variable $((\ln [(\delta - 1)s + 1])/(\delta - 1)) \longrightarrow s$; then, we get the integral involving the Laplace transform.

1. Introduction

Fractional calculus is a fast-growing field of mathematics that shows the relations of fractional-order derivatives and integrals. Fractional calculus is an effectual subject to study many complex real-world systems. In recent years, many researchers have calculated the properties, applications, and extensions of fractional integral and differential operators involving the various special functions.

Fernandez and Baleanu [1] showed that many other named models of fractional calculus can fit within the class of operators defined by Prabhakar and that this class contains both singular and nonsingular operators together. They also characterized completely the cases in which these operators are singular or nonsingular and the cases in which they can be written as finite or infinite sums of Riemann–Liouville differ integrals, to obtain finally a catalogue of subclasses with different types of properties.

Atangana and Baleanu [2] proposed a new fractional derivative with nonlocal and nonsingular kernel. They presented some useful properties of the new derivative and applied it to solve the fractional heat transfer model.

Atangana and Koca [3] presented relationship of derivatives with some integral transform operators. New results are presented. They applied the derivative to a simple nonlinear system. They showed in detail the existence and uniqueness of the system solutions of the fractional system. They obtained a chaotic behavior which was not obtained by local derivative.

Atangana and Baleanu [4] extended the model of the movement of subsurface water via the geological formation called aquifer using a newly proposed derivative with fractional order. An alternative derivative to that of Caputo-Fabrizio with fractional order was presented. The relationship between both derivatives was presented. The new equation was solved analytically using some integral transforms. The exact solution is therefore compared to experimental data obtained from the settlement of the University of the Free State in South Africa. The numerical simulation shows the agreement of the experimental data with an analytical solution for some values of fractional order.

Manzoor et al. [5] used a Beta operator with Caputo (MSM) fractional differentiation of extended Mittag-Leffler function in terms of Beta function. They applied the Beta operator on the right-sided MSM fractional differential operator and on the left-sided MSM fractional differential operator. They also applied the Beta operator on the right-sided MSM fractional differential operator with Mittag-

Leffler function and the left-sided MSM fractional differential operator with Mittag-Leffler function.

Yavuz [6] investigated the novel solutions of fractional-order option pricing models and their fundamental mathematical analyses. The main novelties of this paper are the analysis of the existence and uniqueness of European-type option pricing models providing to give fundamental solutions to them and a discussion of the related analyses by considering both the classical and generalized Mittag-Leffler kernels. Yavuz and Abdeljawad [7] presented a fundamental solution method for nonlinear fractional regularized long-wave (RLW) models. Since analytical methods cannot be applied easily to solve such models, numerical or semianalytical methods have been extensively considered in the literature.

Jena et al. [8] applied two-hybrid techniques, namely, q-homotopy analysis Elzaki transform method (q-HAETM) and iterative Elzaki transform method (IETM) to obtain the numerical solutions of time-fractional Navier-Stokes equations in polar coordinate described in the Caputo sense. q-HAETM is the combination of the homotopy analysis method and Elzaki transform method, and IETM is the combination of two reliable methods, i.e., iterative method and Elzaki transform method.

Yavuz and Sene [9] address the solution of the incompressible second-grade fluid models. Fundamental qualitative properties of the solution are primarily studied for proving the adequacy of the physical interpretations of the proposed model. They used the Liouville-Caputo fractional derivative with its generalized version that gives more comprehensive physical results in the analysis and investigations.

Yavuz [10] analyzed the behaviors of two different fractional derivative operators defined in the last decade. One of them is defined with the normalized sinc function (NSF) and the other one is defined with the Mittag-Leffler function (MLF). Both of them have a nonsingular kernel. Yavuz and Bonyah [11] examined the schistosomiasis fractional-order dynamic model via exponential law kernel sense and Mittag-Leffler kernel in Liouville–Caputo sense. Some special solutions for two operators are obtained using the iterative scheme through Laplace transform and Sumudu–Picard integration technique. The uniqueness and existence of solution for both operators are established. The numerical solutions for both operators approve that the desirable results can be obtained when the alpha value is less than one.

Yang [12] addressed a class of the fractional derivatives of constant and variable orders for the first time. Fractional-order relaxation equations of constants and variable orders in the sense of Caputo type are modeled from a mathematical point of view. The comparative results of the anomalous relaxation among the various fractional derivatives are also given. Yang [13] proposed the general Riemann-Liouville and Caputo-Liouville fractional derivatives with nonsingular power-law kernels, for the first time. New general laws of deformation within the framework of the general fractional derivatives are considered in detail. The creep and relaxation behaviors of the general fractional-order Voigt and Maxwell models are also obtained with the use of the Laplace transform.

The series

$$S(z) = \sum_{m=1}^{\infty} \frac{2m}{\left(m^2 + z^2\right)^2} \quad (1)$$

is known as the Mathieu series. The first person to present such a sequence was Mathieu [14]. Emersleben [15] in elegant form gives an essential meaning of the integral demonstration

$$S(z) = \int_0^{\infty} \frac{x \sin (zx)}{e^x - 1} dx. \quad (2)$$

The above series is also written in the expressions of Riemann-Zeta function by Choi and Srivastava [16] as given below:

$$S(z) = 2 \sum_{m=1}^{\infty} (-1)^m z^{2m} (m+1) \zeta(2m+3), \quad (3)$$

where $|z| <1$ and $\zeta(u) = \sum_{m=1}^{\infty} m^{-u}, u > 1$.

The generalized form of the Mathieu series by Cerone and Lenard [17] is given below as we have

$$S_{\lambda}(z) = \sum_{m=1}^{\infty} \frac{2m}{\left(m^2 + z^2\right)^{\lambda+1}}, \quad (4)$$

where $\lambda > 0$ and $z > 0$; also, it can be written in the expression of Riemann-Zeta function by Pogány et al. [18] as given below:

$$S_{\lambda}(z) = 2 \sum_{m=1}^{\infty} (-1)^m z^{2m} \binom{\lambda + m}{m} \zeta(2\lambda + 2m + 1), \quad (5)$$

where $\lambda > 0$ and $z > 0$.

Taking the above equation in mind, let the p-extended Mathieu series by Pogany and Parmar [19] be well-defined as

$$S_{\lambda,p}(z) = 2 \sum_{m=1}^{\infty} (-1)^m z^{2m} \binom{\lambda + m}{m} \zeta_p(2\lambda + 2m + 1), \quad (6)$$

where $p \geq 0$, $\lambda > 0$, $|z| < 1$, and ζ_p stands for the p-extended Riemann zeta function by Chaudhry et al. [20] which is defined as

$$\zeta_p(u) = \frac{1}{\Gamma(\beta)} \int_0^{\infty} \frac{u^{\beta-1} e^{-(p/u)}}{e^u - 1} dx, \quad (7)$$

where $\Re(p) > 0$ and $\Re(\beta) > 0$.

Shah et al. [21] studied a compartmental mathematical model for the transmission dynamics of the novel coronavirus-19 under Caputo fractional order derivative. By using the fixed-point theory of Schauder and Banach, they established some necessary conditions for existence of at least one solution to model under investigation and its uniqueness. After the existence, a general numerical algorithm based on the Haar collocation method is established to compute the approximate solution of the model.

Sher et al. [22] studied the novel coronavirus (2019-nCoV or COVID-19) which is a threat to the whole world nowadays. They considered a fractional-order epidemic model which describes the dynamics of COVID-19 under nonsingular kernel type of fractional derivative. An attempt is made to discuss the existence of the model using the fixed-point theorem of Banach and Krasnoselskii type.

Definition 1. As indicated in Pohlen [23], let $g(z)=\sum_{m=0}^{\infty}x_m z^m$ and $h(z)=\sum_{m=0}^{\infty}y_m z^m$ be two power series; then, the Hadamard product of power series is defined as

$$(g*h)(z)=\sum_{m=0}^{\infty}x_m y_m z^m=(h.g)(z)\,;\,(|z|<R), \tag{8}$$

where

$$R=\lim_{m\to\infty}\left|\frac{x_m y_m}{x_{m+1}y_{m+1}}\right|\left(\lim_{k\to\infty}\left|\frac{x_m}{x_{m+1}}\right|\right)\cdot\left(\lim_{m\to\infty}\left|\frac{y_m}{y_{m+1}}\right|\right)=R_g\cdot R_h, \tag{9}$$

where R_g and R_h stand for radius of convergence of the above series $g(z)$ and $h(z)$, respectively. Therefore, in general, it is to be noted that if the one power series is the analytical function, then the series of Hadamard products is also the same as an analytical function.

Definition 2. The Gaussian hypergeometric function or ordinary hypergeometric function by Rainville [24] well-defined like ${}_2F_1(a,b\,;c\,;x)$ is the special function which is represented by the hypergeometric series

$${}_2F_1(a,b\,;c\,;x)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{x^m}{m!}, \tag{10}$$

for "c" neither zero nor a negative integer; then, the above notation $(a)_m$ is

$$(a)_m=a(a+1)(a+2)\cdots(a+m-1),\quad m\geq 1. \tag{11}$$

Also, $(a)_0=1$ and $a\neq 0$.

Definition 3. The Laplace transform of the function $f(x)$ on interval $[0,\infty)$ by Sneddon [25] is defined as

$$\mathscr{L}[f(x)\,;s]=\int_0^{\infty}e^{-sx}f(x)dx=F(s), \tag{12}$$

where $s\in C$ and $x\geq 0$.

Definition 4. The Elzaki transform for the function of exponential order by Elzaki [26] is considered the function in the set Y; we get

$$Y=\left\{f(x)\colon \exists W,n_1,n_2>0|f(x)|<\mathrm{W}e^{|x|/n_j},x\in(-1)^jA[0,\infty)\right\}. \tag{13}$$

For the function which is constant in set Y, W, it must be a finite number, so maybe n_1 and n_2 are finite or infinite at that time the Elzaki transform denoted through operator E as given below:

$$E[f(x)]=T(u)=u\int_0^{\infty}f(x)e^{x/u}dx, \tag{14}$$

where $n_1\leq u\leq n_2$.

Definition 5. Let the function $f(x)$ be integrable with a finite interval, (n_1,n_2), $(n_1<x<n_2)$; if there exists a real number "r," then each of the following statements holds true, so, as$n_1>0$, $\int_{\lambda}^{n_1}|f(x)|dx$ approaches to a finite limit like $\lambda\longrightarrow 0+$; also, here, $n_2>0$, $\int_{n_2}^{\hbar}e^{-rx}f(x)dx$ approaches to a finite limit like $\hbar\longrightarrow\infty$; then, the P_δ-transform,

$$G_{P_\delta}(s)=P_\delta[f(x)\,;s]=\int_0^{\infty}[(\delta-1)s+1]^{x/\delta-1}f(x)dx,\delta>1, \tag{15}$$

exists whenever $\Re(\ln[1+(\delta-1)s]/(\delta-1))>r$, $s\in C$.

The power function of the transform by Kumar [27] and Nadir and Khan [28] is given below:

$$P_\delta\left[x^{\eta-1}\,;s\right]=\left(\frac{\delta-1}{\ln[1+(\delta-1)s]}\right)^{\eta}\Gamma(\eta),\quad \Re(\eta)>0,\delta>0. \tag{16}$$

2. The P_δ-Transform Associated with the p-Extended Mathieu Series

Here, we have evaluated the P_δ-transform associated with the p-extended Mathieu series and some of its certain cases in the form of corollaries.

Theorem 6. *Let the p-extended Mathieu series be given in (6) as*

$$S_{\lambda,p}(z)=2\sum_{m=1}^{\infty}(-1)^m z^{2m}\binom{\lambda+m}{m}\zeta_p(2\lambda+2m+1), \tag{17}$$

where $p\geq 0$, $\lambda>0$, and $|z|<1$ and ζ_p is known as the p-extended Riemann zeta function. Now, by applying the P_δ-

transform on the p-extended Mathieu series, we want to show that

$$P_\delta\left[t^{\sigma-1}S_{\lambda,p}(z,t)\,;s\right]=\frac{\Gamma(\sigma)}{[\varphi(\delta,s)]^{\sigma}}\,S_{\lambda,p}\left(\frac{4z^2}{[\varphi(\delta,s)]^2}\right) * F_0^2\left[\begin{matrix}\frac{\sigma}{2},\frac{\sigma+1}{2}\,;\frac{4z^2}{[\varphi(\delta,s)]^2}\\ -;\end{matrix}\right], \quad (18)$$

where $\min\{\Re(s),\Re(\sigma)\}>0$, $[\varphi(\delta,s)]=\ln[(\delta-1)s+1]/(\delta-1)$, $|z|<1$, $\delta>1$, *and* F_0^2 *is the Gaussian hypergeometric function as defined in the book of special function by Rainville [24].*

Proof. Consider

$$\begin{aligned}\text{L.H.S}&=P_\delta\left[t^{\sigma-1}S_{\lambda,p}(z,t)\,;s\right]\\&=\int_0^\infty t^{\sigma-1}[(\delta-1)s+1]^{-t/\delta-1}S_{\lambda,p}(z,t)dt\\&=\int_0^\infty t^{\sigma-1}[(\delta-1)s+1]^{-t/\delta-1}2\\&\quad\cdot\sum_{m=1}^{\infty}(-1)^m(z)^{2m}\begin{pmatrix}\lambda+m\\m\end{pmatrix}\zeta_p(2\lambda+2m+1)dt.\end{aligned} \quad (19)$$

Due to uniform convergence, we have changed the order of integration and summation:

$$\begin{aligned}\text{L.H.S}&=2\sum_{m=1}^{\infty}(-1)^m\begin{pmatrix}\lambda+m\\m\end{pmatrix}\zeta_p(2\lambda+2m+1)(z)^{2m}\\&\quad\cdot\int_0^\infty t^{\sigma-1}[(\delta-1)s+1]^{-t/\delta-1}.\end{aligned} \quad (20)$$

Now, changing σ by $\sigma+2m$, we get

$$\begin{aligned}\text{L.H.S}&=2\sum_{m=1}^{\infty}(-1)^m\begin{pmatrix}\lambda+m\\m\end{pmatrix}\zeta_p(2\lambda+2m+1)(z)^{2m}\\&\quad\times\int_0^\infty t^{\sigma+2m-1}e^{\ln\,[(\delta-1)s+1]^{-t/\delta-1}}dt.\end{aligned} \quad (21)$$

Here, by using equation (16), we get

$$\begin{aligned}\text{L.H.S}&=2\sum_{m=1}^{\infty}(-1)^m\begin{pmatrix}\lambda+m\\m\end{pmatrix}\zeta_p(2\lambda+2m+1)(z)^{2m}\\&\quad\cdot\left(\frac{\delta-1}{\ln[(\delta-1)+1]s}\right)^{\sigma+2m}\Gamma(\sigma+2m).\end{aligned} \quad (22)$$

Also, using (18), we have

$$\begin{aligned}\text{L.H.S}&=2\sum_{m=1}^{\infty}(-1)^m\begin{pmatrix}\lambda+m\\m\end{pmatrix}\zeta_p(2\lambda+2m+1)(z)^{2m}\\&\quad\cdot\left(\frac{1}{\varphi[\delta,s]}\right)^{\sigma+2m}\Gamma(\sigma+2m).\end{aligned} \quad (23)$$

By Rainville [24], we have

$$\begin{aligned}\text{L.H.S}&=\frac{\Gamma(\sigma)}{[\varphi(\delta,s)]^{\sigma}}2\sum_{m=1}^{\infty}(-1)^m\begin{pmatrix}\lambda+m\\m\end{pmatrix}\zeta_p(2\lambda+2m+1)\\&\quad\times\left(\frac{z}{[\varphi(\delta,s)]}\right)^{2m}2^{2m}\left(\frac{\sigma}{2}\right)_m\left(\frac{\sigma+1}{2}\right)_m.\end{aligned} \quad (24)$$

Now, by using (6) and (10), we have

$$\text{L.H.S}=\frac{\Gamma(\sigma)}{[\varphi(\delta,s)]^{\sigma}}\,S_{\lambda,p}\left(\frac{4z^2}{[\varphi(\delta,s)]^2}\right) * F_0^2\left[\begin{matrix}\frac{\sigma}{2},\frac{\sigma+1}{2}\,;\frac{4z^2}{[\varphi(\delta,s)]^2}\\ -;\end{matrix}\right]. \quad (25)$$

So, here, we have proved

$$P_\delta\left[t^{\sigma-1}S_{\lambda,p}(z,t)\,;s\right]=\frac{\Gamma(\sigma)}{[\varphi(\delta,s)]^{\sigma}}\,S_{\lambda,p}\left(\frac{4z^2}{[\varphi(\delta,s)]^2}\right) * F_0^2\left[\begin{matrix}\frac{\sigma}{2},\frac{\sigma+1}{2}\,;\frac{4z^2}{[\varphi(\delta,s)]^2}\\ -;\end{matrix}\right], \quad (26)$$

where $\min\{\Re(s),\Re(\sigma)\}>0$, $[\varphi(\delta,s)]=\ln[(\delta-1)s+1]/(\delta-1)$, $|z|<1$, $\delta>1$, and F_0^2 is the Gaussian hypergeometric function.

Corollary 7. *Let* $p=0$, $\lambda>0$, *and* $|z|<1$; *then, the following relation holds true; we have*

$$P_\delta\left[t^{\sigma-1}S_{\lambda}(z,t)\,;s\right]=\frac{\Gamma(\sigma)}{[\varphi(\delta,s)]^{\sigma}}\,S_{\lambda}\left(\frac{4z^2}{[\varphi(\delta,s)]^2}\right) * F_0^2\left[\begin{matrix}\frac{\sigma}{2},\frac{\sigma+1}{2}\,;\frac{4z^2}{[\varphi(\delta,s)]^2}\\ -;\end{matrix}\right], \quad (27)$$

where $\min\{\Re(s),\Re(\sigma)\}>0$, $[\varphi(\delta,s)]=\ln[(\delta-1)s+1]/(\delta-1)$, $\delta>1$, *and* F_0^2 *is the Gaussian hypergeometric function.*

Corollary 8. *Let $p = 0$, $\lambda = 1$, and $|z| < 1$; then, the following relation also holds true:*

$$P_\delta\left[t^{\sigma-1}S(z,t)\,;s\right] = \frac{\Gamma(\sigma)}{[\varphi(\delta,s)]^\sigma}\,S\left(\frac{4z^2}{[\varphi(\delta,s)]^2}\right) * F_0^2\left[\begin{matrix}\dfrac{\sigma}{2},\dfrac{\sigma+1}{2};\dfrac{4z^2}{[\varphi(\delta,s)]^2}\\ -;\end{matrix}\right], \tag{28}$$

where $\min\{\Re(s),\Re(\sigma)\} > 0$, $[\varphi(\delta,s)] = \ln[(\delta-1)s+1]/(\delta-1)$, $\delta > 1$, *and* F_0^2 *is the Gaussian hypergeometric function.*

Remark 9. It is to be noted that

$$S_{\lambda,p}(z) = 2\sum_{m=1}^{\infty}(-1)^m z^{2m}\begin{pmatrix}\lambda+m\\ m\end{pmatrix}\zeta_p(2\lambda+2m+1) \tag{29}$$

reduces to

$$S_\lambda(z) = 2\sum_{m=1}^{\infty}(-1)^m z^{2m}\begin{pmatrix}\lambda+m\\ m\end{pmatrix}\zeta(2\lambda+2m+1), \tag{30}$$

when $p = 0$, further for $\lambda = 1$; in the above equations, we have the following relation, $S(z) = 2\sum_{m=1}^{\infty}(-1)^m z^{2m}(m+1)\zeta(2m+3)$. So, here, we have seen that the P_δ-transform holds true for the p-extended Mathieu series, the generalized Mathieu series, and also for the Mathieu series.

3. Special Cases

Here, we have converted the P_δ-transform into a classical Laplace transform by changing the variable $(\ln[(\delta-1)s+1]/(\delta-1)) \longrightarrow s$; then, we get integral involving Laplace transform as given below.

Corollary 10. *Let $p \geq 0$, $\lambda > 0$, $|z| < 1$, and $\Re(s) > 0$; then, the Laplace transform formula holds true and establishes the following result:*

$$\mathscr{L}\left[t^{\sigma-1}S_{\lambda,p}(z,t)\,;s\right] = \frac{\Gamma(\sigma)}{s^\sigma}\,S_{\lambda,p}\left(\frac{4z^2}{s^2}\right) * F_0^2\left[\begin{matrix}\dfrac{\sigma}{2},\dfrac{\sigma+1}{2};\dfrac{4z^2}{s^2}\\ -;\end{matrix}\right]. \tag{31}$$

Proof.

$$\begin{aligned}\text{L.H.S} &= \mathscr{L}\left[t^{\sigma-1}S_{\lambda,p}(z,t)\,;s\right]\\ &= \int_0^\infty t^{\sigma-1}\cdot e^{-st}\left(2\sum_{m=1}^{\infty}(-1)^m z^{2m}\begin{pmatrix}\lambda+m\\ m\end{pmatrix}\zeta_p(2\lambda+2m+1)\right)dt.\end{aligned} \tag{32}$$

Due to uniform convergence, we have changed the order of integration and summation:

$$\text{L.H.S} = \left(2\sum_{m=1}^{\infty}(-1)^m z^{2m}\begin{pmatrix}\lambda+m\\ m\end{pmatrix}\zeta_p(2\lambda+2m+1)\right)\cdot\int_0^\infty t^{\sigma-1}e^{-st}dt. \tag{33}$$

Now, by changing σ to $\sigma + 2m$, we get

$$\text{L.H.S} = \left(2\sum_{m=1}^{\infty}(-1)^m\begin{pmatrix}\lambda+m\\ m\end{pmatrix}\zeta_p(2\lambda+2m+1)z^{2m}\right)\cdot\int_0^\infty t^{\sigma+2m-1}e^{-st}dt. \tag{34}$$

Using equation (16), we get

$$\text{L.H.S} = \left(2\sum_{m=1}^{\infty}(-1)^m\begin{pmatrix}\lambda+m\\ m\end{pmatrix}\zeta_p(2\lambda+2m+1)\right)\cdot z^{2m}\frac{\Gamma(\sigma+2m)}{s^{\sigma+2m}}. \tag{35}$$

Now, by Rainville [24], we get

$$\text{L.H.S} = \frac{\Gamma(\sigma)}{s^\sigma}\left(2\sum_{m=1}^{\infty}(-1)^m\begin{pmatrix}\lambda+m\\ m\end{pmatrix}\zeta_p(2\lambda+2m+1)\right)\cdot\left(\frac{z}{s}\right)^{2m}2^{2m}\left(\frac{\sigma}{2}\right)_m\left(\frac{\sigma+1}{2}\right)_m. \tag{36}$$

Using (6) and (10), we obtained

$$\text{L.H.S} = \frac{\Gamma(\sigma)}{s^\sigma}S_{\lambda,p}\left(\frac{4z^2}{s^2}\right) * F_0^2\left[\begin{matrix}\dfrac{\sigma}{2},\dfrac{\sigma+1}{2};\dfrac{4z^2}{s^2}\\ -;\end{matrix}\right]. \tag{37}$$

Hence, we arrived at the required result

$$\mathscr{L}\left[t^{\sigma-1}S_{\lambda,p}(z,t)\,;s\right] = \frac{\Gamma(\sigma)}{s^\sigma}\,S_{\lambda,p}\left(\frac{4z^2}{s^2}\right) * F_0^2\left[\begin{matrix}\dfrac{\sigma}{2},\dfrac{\sigma+1}{2};\dfrac{4z^2}{s^2}\\ -;\end{matrix}\right], \tag{38}$$

where $p \geq 0$, $\lambda > 0$, and $|z| < 1$, and $\Re(s) > 0$.

Corollary 11. *Let $p = 0$, $\lambda > 0$, $|z| < 1$, and $\Re(s) > 0$; then, the Laplace transform formula holds true and we have*

$$\mathscr{L}\left[t^{\sigma-1}S_\lambda(z,t)\,;s\right] = \frac{\Gamma(\sigma)}{s^\sigma}\,S_\lambda\left(\frac{4z^2}{s^2}\right) * F_0^2\left[\begin{matrix}\frac{\sigma}{2},\frac{\sigma+1}{2};\frac{4z^2}{s^2}\\ -;\end{matrix}\right]. \tag{39}$$

Corollary 12. *Let $p = 0$, $\lambda = 1$, $|z| < 1$, and $\Re(s) > 0$; then, the Laplace transform formula holds true as*

$$\mathscr{L}\left[t^{\sigma-1}S(z,t)\,;s\right] = \frac{\Gamma(\sigma)}{s^\sigma}\,S\left(\frac{4z^2}{s^2}\right) * F_0^2\left[\begin{matrix}\frac{\sigma}{2},\frac{\sigma+1}{2};\frac{4z^2}{s^2}\\ -;\end{matrix}\right]. \tag{40}$$

Remark 13. Thus, it is to be prominent that the P_δ-transform is immediately reduced to the Elzaki transform by shifting the variable ln $[(\delta-1)s+1]/(\delta-1)$ into s.

4. Conclusion

It is noted that the $\underline{p}$-extended Mathieu series is more general in nature and various generalized types of Mathieu series defined in literature can easily be derived through the extended form. Similarly, the P_δ-transform defined by Kumar's [27] fractional integral operator assists us in converting the table of the Laplace transform and the Elzaki transform into the corresponding transform and vice versa.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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